1. (10)

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} t \\ 2\ln t \\ 1-\frac{2}{t} \end{pmatrix}, \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2}{t} \\ \frac{2}{t^2} \end{pmatrix}$$

$$\ell = \int_1^2 \sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt = \int_1^2 \sqrt{1^2+\frac{4}{t^2}+\frac{4}{t^4}}\,dt$$

$$= \int_1^2 \sqrt{\left(1+\frac{2}{t^2}\right)^2}\,dt = \int_1^2 \left(1+\frac{2}{t^2}\right)dt = t-\frac{2}{t}\Big|_1^2 = 2-1-1+2$$
$$= 2$$

2. (10) Compute the cross product of

$$\bar{a} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \text{ and } \bar{b} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$

$$\bar{a}\times\bar{b} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 3 & 2 \\ -1 & 0 & -2 \end{vmatrix} = e_1(-6) - e_2\cdot 2 + e_3\cdot 3$$
$$= \begin{pmatrix} -6 \\ -2 \\ 3 \end{pmatrix}$$

Normalize $|\bar{a}\times\bar{b}| = \sqrt{36+4+9} = \sqrt{49} = 7$

Two vectors: $\begin{pmatrix} -6/7 \\ -2/7 \\ 3/7 \end{pmatrix}$ and $\begin{pmatrix} 6/7 \\ 2/7 \\ -3/7 \end{pmatrix}$.

3. $f(x,y) = \frac{e^y}{x}$. Taylor expansion up to order 2 at $(x,y) = (1,0)$.

$$f_x(x,y) = -\frac{e^y}{x^2}, \quad f_y(x,y) = \frac{e^y}{x}$$

$$f_{xx}(x,y) = \frac{2e^y}{x^3}, \quad f_{xy}(x,y) = f_{yx}(x,y) = -\frac{e^y}{x^2}, \quad f_{yy}(x,y) = \frac{e^y}{x}$$

$$f(1,0) = \frac{1}{1} = 1$$

$f_x(1,0) = -1, \quad f_y(1,0) = 1$

$f_{xx}(1,0) = 2, \quad f_{xy}(1,0) = f_{yx}(1,0) = -1, \quad f_{yy}(1,0) = 1$

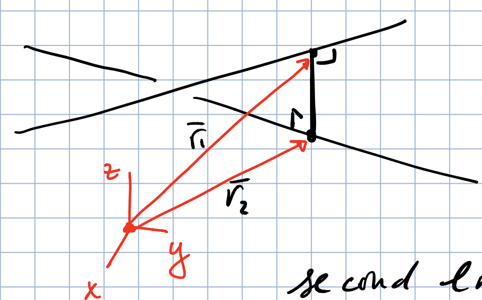
$$f(x,y) = 1 - (x-1) + y + \frac{2}{2!}(x-1)^2 - (x-1)y + \frac{1}{2!}y^2 + R_2(x,y)$$

Taylor polynomial: $2 - x + y + (x-1)^2 - (x-1)y + \frac{1}{2}y^2$

Further simplification: $3 - 3x + 2y + x^2 - xy + \frac{1}{2}y^2$

4.

10




first line:

$x = -2z + 4$

$y - 2 = -2z + 4$

$y = -2z + 6$

$$\bar{r}_1(t) = \begin{pmatrix} -2t+4 \\ -2t+6 \\ t \end{pmatrix}$$

second line:

$z - 1 = -2y + 6 \quad z = -2y + 7$

$x - 3 = -2y + 6 \quad x = -2y + 9$

$$\bar{r}_2(s) = \begin{pmatrix} -2s+9 \\ s \\ -2s+7 \end{pmatrix}$$

$$\bar{r}_1(t) - \bar{r}_2(s) \perp \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \text{ and } \perp \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$$

$$\Longrightarrow \bar{r}_1(t) - \bar{r}_2(s) = \begin{pmatrix} -2t+2s-5 \\ -2t-s+6 \\ t+2s-7 \end{pmatrix}$$

$$\begin{pmatrix} -2t+2s-5 \\ -2t-s+6 \\ t+2s-7 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} -2t+2s-5 \\ -2t-s+6 \\ t+2s-7 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = 0$$

$9t + 10 - 12 - 7 = 0$

$9t - 9 = 0 \Rightarrow t = 1$

$-9s + 10 + 6 + 14 = 0$

$-9s + 30 = 0$

$s = \frac{10}{3}$

$$\bar{r}_1(1) - \bar{r}_2\left(\frac{10}{3}\right) = \begin{pmatrix} -7 + \frac{20}{3} \\ 4 - \frac{10}{3} \\ -6 + \frac{20}{3} \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

length: $\ell = 1/3 \cdot \sqrt{1+4+4} = 1/3 \sqrt{9} = 1.$

5. 10

$$z = f(x,y) = x^y + y = e^{y \ln x} + y$$

$$x = u^2 + v^2 - 4, \quad y = uv - 1$$

Chain rul: $\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$

$$\frac{\partial x}{\partial u} = 2u, \quad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial f}{\partial x} = e^{y \ln x} \cdot \frac{y}{x}, \quad \frac{\partial f}{\partial y} = e^{y \ln x} \cdot \ln x + 1$$

$$\frac{\partial z}{\partial u} = e^{y \ln x} \frac{y}{x} \cdot 2u + \left(e^{y \ln x} \ln x + 1\right) v$$

$$u = 2, \quad v = -1, \quad x = 4 + 1 - 4 = 1, \quad y = 2 \cdot (-1) - 1 = -3$$

$$\frac{\partial z}{\partial u}(2,-1) = e^{-3 \cdot 0} \cdot \frac{-3}{1} \cdot 2 \cdot 2 + \left(e^{-3 \cdot 0} \cdot 0 + 1\right) \cdot (-1)$$

$$= -3 \cdot 4 - 1 = -13$$

6. Define $L(x,y,\lambda) = xy^2 + \lambda(x^2 + y^2 - 3)$

10

(1) $\partial_x L = y^2 + 2x\lambda = 0 \qquad x$

(2) $\partial_y L = 2xy + 2y\lambda = 0 \qquad y$

(3) $\partial_\lambda L = x^2 + y^2 - 3 = 0$

$(1) \cdot x + (2) \cdot y = 0$

$$xy^2 + 2xy^2 + (2x^2 + 2y^2)\lambda = 0 \qquad \text{use } (3)$$

$$3xy^2 + 6\lambda = 0 \qquad \lambda = -\tfrac{1}{2}xy^2$$

Substitute in (1) $\quad y^2 - x^2y^2 = 0$

$$(1-x^2)y^2 = 0$$

" " (2) $\quad 2xy - xy^3 = 0$

$$xy(2-y^2) = 0$$

$y = 0 \implies x = \pm\sqrt{3}$

$x = 0 \implies y = 0$ Not possible since $x^2+y^2=3$

$x = \pm 1 \implies y = \pm\sqrt{2}$

Last option $\quad (1, \pm\sqrt{2}),\ f = 1\cdot(\pm\sqrt{2})^2 = 2$

$(-1, \pm\sqrt{2}),\ f = -1\cdot(\pm\sqrt{2})^2 = -2$

For $(\pm\sqrt{3}, 0)$, $f = 0$ so <u>not</u> a <u>max</u> or <u>min</u>.

Maximal value of $f = 2$

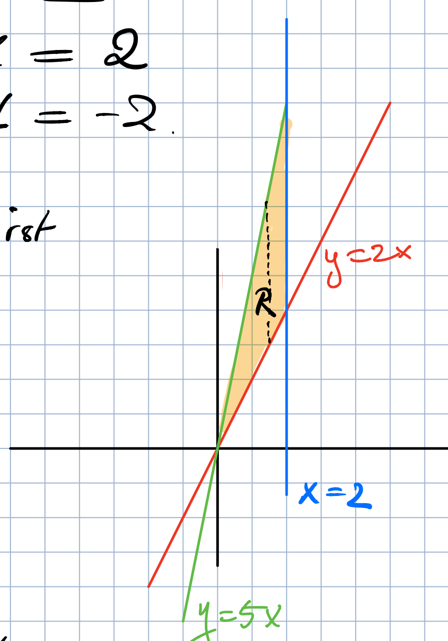
Minimal value of $f = -2$.

7. (10) Integrate in $y$ first

$$\int_{x=0}^{x=2}\int_{y=2x}^{y=5x} y^2\,dy\,dx$$

$$= \int_0^2 \tfrac{1}{3}y^3\Big|_{2x}^{5x}\,dx$$

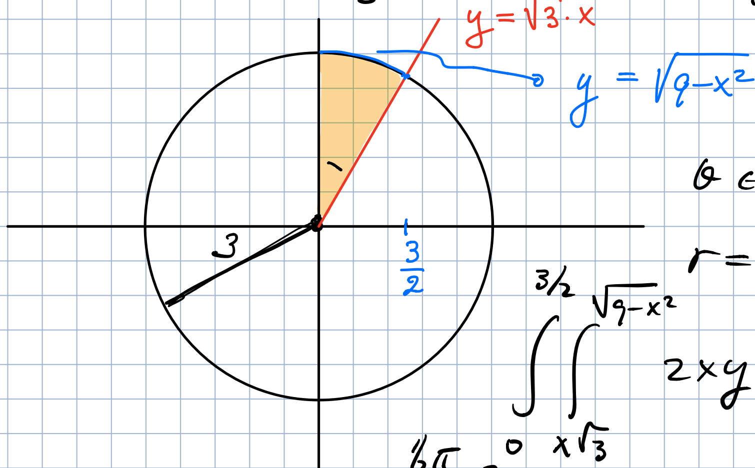
$$= \int_0^2 \tfrac{1}{3}(125-8)x^3\,dx$$



$$= \int_0^2 \frac{117}{3} x^3\,dx = \int_0^2 39x^3\,dx = \frac{39}{4} x^4 \Big|_0^2 = 39\cdot 4 = 156$$

8.

15




$\theta \in [\frac{1}{3}\pi, \frac{1}{2}\pi]$

$r = [0, 3]$

$$\int_0^{3/2}\int_{x\sqrt{3}}^{\sqrt{9-x^2}} 2xy\,dy\,dx$$

$$= \int_{\frac13\pi}^{\frac12\pi}\int_0^3 2r^2\cos\theta\sin\theta\, r\,dr\,d\theta$$

$$= \int_{\frac13\pi}^{\frac12\pi}\int_0^3 r^3 \sin 2\theta\,dr\,d\theta = \frac14 r^4\Big|_0^3 \cdot \int_{\frac13\pi}^{\frac12\pi}\sin 2\theta\,d\theta$$

$$= \frac{81}{4}\cdot\frac{-1}{2}\cos(2\theta)\Big|_{\frac13\pi}^{\frac12\pi} = -\frac{81}{8}\left(\cos\pi - \cos(\tfrac23\pi)\right)$$

$$= -\frac{81}{8}\left(-1+\frac12\right) = \frac{81}{16}$$

9.

15

$$\iiint_E 15x^2\,dx\,dy\,dz = \int_{y=0}^{y=2}\int_{z=0}^{z=2-y}\int_{x=0}^{x=2-y-z} 15x^2\,dx\,dz\,dy$$

$$= \int_0^2\int_0^{2-y} 5x^3\Big|_0^{2-y-z} dz\,dy = 5\int_0^2\int_0^{2-y}(2-y-z)^3\,dz\,dy$$

$$= -\frac54\int_0^2 (2-y-z)^4\Big|_0^{2-y} dy = +\frac54\int_0^2 (2-y)^4\,dy$$

$$= -\frac{1}{4}(2-y)^5\Big|_0^2 = \frac{1}{4}\cdot 2^5 = 2^3 = 8$$