Exam Measure Theory

October 21, 2015

Give crisp and clear argumentations.

This exam consists of four exercises with, in total, nine parts. In the grading, all these nine parts will be weighted equally.

1. Let X be a set and $\mathcal{F}$ a family of subsets of X with the following three properties below:

$$\emptyset \in \mathcal{F},$$

$$A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F},$$

$$((A_j)_{j\in\mathbb{N}} \subset \mathcal{F} \text{ and } A_1 \subset A_2 \subset A_3 \subset \cdots) \Longrightarrow \bigcup_{j\in\mathbb{N}} A_j \in \mathcal{F}.$$

Show that $\mathcal{F}$ is a σ-algebra.

2. Consider a measure space $(X, \mathcal{A}, \mu)$, with $\mu(X)$ finite.
Let $A_1, \cdots, A_m \in \mathcal{A}$. Let n be a positive integer $\leq m$. Let V_n be the set of all $x \in X$ which belong to at least n of the sets $A_1, \cdots, A_m$.
(a) Show that V_n is measurable, i.e. that $V_n \in \mathcal{A}$.
(b) Show that if $V_n = X$, then there is a $j \in \{1, \cdots, m\}$ with $\mu(A_j) \geq \frac{\mu(X)\,n}{m}$.
(c) Use the result in (b) to show that, in the more general case where V_n may be different from X, there is a $j \in \{1, \cdots, m\}$ with $\mu(A_j) \geq \frac{\mu(V_n)\,n}{m}$.

3. Let $\mathcal{J}$ be the family of all infinite subsets of $\mathbb{Q}$.
(a) Show that the σ-algebra on $\mathbb{Q}$ generated by $\mathcal{J}$ is the family of *all* subsets of $\mathbb{Q}$ (which we will denote by $\mathcal{P}$).
(b) Let μ be the counting measure on $\mathcal{P}$ (i.e. $\mu(A) = \infty$ if A is infinite and $\mu(A)$ is equal to the number of elements of A if A is finite). Does there exist a measure $\nu \neq \mu$ on $\mathcal{P}$ with the property that $\nu(A) = \mu(A)$ for all $A \in \mathcal{J}$?

4. Let $f : \mathbb{R} \to \mathbb{R}^2$ be the function $f(x) = (x, x)$.
(a) Show that f is $\mathcal{B}(\mathbb{R})\,/\mathcal{B}(\mathbb{R}^2)$ - measurable.
(b) Let V be a non-Borel measurable subset of $\mathbb{R}$. Show that the set $A := \{(x, x) : x \in V\}$ is not Borel measurable.
(c) Show that A is in the completion of $\mathcal{B}(\mathbb{R}^2)$.