Exam Measure Theory

February 11, 2015, 18.30-21.15.

1. Let $A \subset \mathbb{R}$ be a measurable set with $\lambda(A) < \infty$, where λ denotes Lebesgue measure. Show that for any $\epsilon > 0$, there exists a bounded set $B \subset A$ such that $\lambda(A \backslash B) < \epsilon$. (A set is called bounded if it is contained in an interval of the form $[-M, M]$, for some $0 < M < \infty$.)

2. Let $X = [0,1] \times \{0,1\}$. Let $\mathcal{A}$ be the collection of all sets $E \subset X$ such that the sections $E_x = \{y : (x,y) \in E\}$ are either empty or coincide with $\{0,1\}$ for all x, except possibly for countably many points x.
(a) Show that $\mathcal{A}$ is a sigma-algebra.

Let μ be the function that assigns to every set $E \in \mathcal{A}$ the cardinality of the intersection of E with $[0,1] \times \{0\}$.
(b) Show that μ is a measure on $\mathcal{A}$.

3. (a) Let $A \subset [0,1]$ be a measurable set with the following properties:

1. $0 < \lambda(A) < 1$;
2. $\lambda(A \cap J) > 0$ for all open intervals $J \subset [0,1]$.

Let f be the indicator function of A. We divide $[0,1]$ into n intervals $I_1, I_2, \ldots, I_n$ of length $1/n$ each. With this partition, we can compute upper and lower Riemann sums, denoted by U_n and L_n respectively. (These are the sums that we use to bound the Riemann integral from above and below respectively.)
(a) Show that $U_n = 1$ and that $L_n \leq \lambda(A)$.
(b) Is f Riemann integrable? Is f Lebesgue integrable? Explain your answers.
(c) Construct an example of a set with properties 1. and 2. above.

4. A measure μ on $(\Omega, \mathcal{F})$ is called *atom free* if $\mu(A) > 0$ implies that there exists $B \subset A$ such that $0 < \mu(B) < \mu(A)$. (A and B are elements of $\mathcal{F}$.) No let λ be Lebesgue measure on $[0,1]$ and $A \subset [0,1]$ be measurable such that $\lambda(A) > 0$. Define $f : [0,1] \to [0,1]$ as

$$f(x) = \lambda(A \cap [0,x]).$$

(a) Show that f is continuous.
(b) Show that λ is atom free.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function.
(a) Show that the graph of f, that is, the set $\{(x, y) : y = f(x)\}$, is a measurable set in the product space $\mathbb{R} \times \mathbb{R}$.
(b) Use Fubini's theorem to prove that the graph has two-dimensional Lebesgue measure zero.

6. 1. Let μ be a finite measure on $(\Omega, \mathcal{F})$. Consider measurable functions $f_1, f_2, \ldots$ on Ω such that $|f_n(x)| \leq C$ for all n and all $x \in \Omega$, where $C < \infty$. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ μ-a.e. Show that

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu.$$