Exam Measure Theory

October 24 2014, 8.45-11.30

Alle onderdelen tellen even zwaar mee.

Exercise 1. Let f be a function from $(\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, where $\mathcal{F}$ is a sigma-algebra, and $\mathcal{B}$ denotes the Borel sigma-algebra.
(a) Does there exist a sigma-algebra $\mathcal{F}$ such that f is $\mathcal{F}/\mathcal{B}$ measurable if and only f is a constant function? Explain your answer.
(b) Does there exist is sigma-algebra $\mathcal{F}$ such that f is $\mathcal{F}/\mathcal{B}$ measurable if and only f is continuous? Explain your answer.

Exercise 2. For μ a pre-measure on a semi-ring $\mathcal{S}$ of subsets of a set X, we have defined the outer measure μ^* as follows:

$$\mu^*(A) := \inf \left\{ \sum_{j=1}^{\infty} \mu(S_i) : S_j \in \mathcal{S}, \bigcup_{j=1}^{\infty} S_j \supset A \right\}.$$

(a) Show that in case $X = \mathbb{R}$, the collection S of intervals of the form $[a, b), a, b \in \mathbb{R}$ is a semi-ring.
(b) We have shown in class that Lebesgue measure λ is a pre-measure on the semi-ring in (a). Show that in this case,

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(S_i) : S_j \text{ is open}, \bigcup_{j=1}^{\infty} S_j \supset A \right\}.$$

You can use the fact that open sets in $\mathbb{R}$ are countable unions of open intervals.

Exercise 3. Let $x \in [0, 1]$ and write x in binary representation as

$$x = 0.a_1(x)a_2(x)a_3(x)\dots$$

where $a_n(x) \in \{0, 1\}$. By this we mean that

$$x = \sum_{n=1}^{\infty} \frac{a_n(x)}{2^n}.$$

(Some x have two such representations. In those cases we choose the one for which $a_n(x) = 1$ for all large enough n.) Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = 0.a_2(x)a_3(x)a_4(x)\ldots = \sum_{n=1}^{\infty} \frac{a_{n+1}(x)}{2^n}.$$

(a) Show that for all n, $a_n : [0,1] \to [0,1]$ is $\mathcal{B}/\mathcal{B}$ measurable.
(b) Let, for $k = 1, 2, \ldots$

$$f_k(x) = \sum_{n=1}^{k} \frac{a_{n+1}(x)}{2^n}.$$

Show that f_k is measurable and use this to show that f is measurable.
(c) Show that f is piecewise continuous.
(d) Use (c) to give a second proof of the measurability of f.

Exercise 4. Let $A_1, A_2, \ldots$ be elements of $\mathcal{B}$ such that $\lambda(A_m \cap A_n) = 0$ for all $m \neq n$. Let $B_n = A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c$.
(a) Show that

$$\lambda(\bigcup_{n=1}^{\infty} A_n) = \lambda(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \lambda(B_n).$$

(b) Show that

$$A_n \triangle B_n \subset \bigcup_{m=1}^{n} (A_m \cap A_n).$$

(Here $\triangle$ denotes symmetric difference.)
(c) Show that

$$\lambda(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \lambda(A_n).$$

Exercise 5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and define the sets $A_k \subset \mathbb{N}$ by

$$A_k = \{k, 2k, 3k, \ldots\},$$

for $k = 1, 2, \ldots$ We denote by $\mathcal{H}$ the collection $\{A_1, A_2, \ldots\} \cup \emptyset$.
(a) Show that $\sigma(\mathcal{H})$ (the sigma-algebra generated by $\mathcal{H}$) is equal to $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$).
(b) Suppose that μ and ν are finite measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ such that $\mu(H) = \nu(H)$ for all $H \in \mathcal{H}$. Show that $\mu = \nu$.