Exam Measure Theory

14 december 2009; 8.45-11.30

All parts have equal weight.

1. Let μ be a finite measure on $(\Omega, \mathcal{F})$. Consider a sequence $f_1, f_2, \ldots$ of measurable functions from $\Omega \to \mathbb{R}$ such that $|f_n(x)| \leq C$ for all n and all $x \in \Omega$, where C is a constant. Supppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for μ-almost all $x \in \Omega$.
(a) Use one of the convergence theorems to show that

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu.$$

(b) Show by example that the conclusion need not be true if μ is not a finite measure.

2. Consider the interval $[0,1]$ with sigma-algebra $\mathcal{G}$ consisting of all subsets of $[0,1]$. Let μ be the measure on $\mathcal{G}$ defined by $\mu(G) = 0$ if G is countable, and $\mu(G) = \infty$ if G is uncountable.
(a) Which functions $f : [0,1] \to \mathbb{R}$ are measurable w.r.t. $\mathcal{G}$ and $\mathcal{B}$, that is, satisfy $f^{-1}(B) \in \mathcal{G}$ for all $B \in \mathcal{B}$?
(b) Give an example of sets $A_1 \supset A_2 \supset \cdots$ with $\cap_{n=1}^\infty A_n = A$ for which $\mu(A_n)$ does not converge to $\mu(A)$ as $n \to \infty$.
(c) Can you give an example of sets $B_1 \subset B_2 \subset \cdots$ with $\cup_{n=1}^\infty B_n = B$ for which $\mu(B_n)$ does not converge to $\mu(B)$ as $n \to \infty$? Motivate your answer.
(d) Consider the restriction μ' of μ to the Borel sigma-algebra $\mathcal{B}$. Show that for Lebesgue measure m on $[0,1]$ we have $m \ll \mu'$.

3. A measure μ on $(\Omega, \mathcal{F})$ is called *nonatomic* if $\mu(A) > 0$ implies that there exists $B \subset A$ such that $0 < \mu(B) < \mu(A)$ (A and B are elements of $\mathcal{F}$).
(a) Give an example of a measure which is not nonatomic.

Let m be Lebesgue measure on $[0,1]$, and $A \subset [0,1]$ measurable with $m(A) > 0$. Define $f : [0,1] \to [0,1]$ by

$$f(x) = m(A \cap [0,x]).$$

(b) Show that f is continuous.
(c) Show, using (b), that m is nonatomic.

4. Let Ω_1 be a space with sigma-algebra $\mathcal{F}_1$, and Ω_2 a space with sigma-algebra $\mathcal{F}_2$. Let $T : \Omega_1 \longrightarrow \Omega_2$ be measurable and let μ_1 be a measure on $\mathcal{F}_1$. Define, for all $E \in \mathcal{F}_2$,

$$\mu_2(E) = \mu_1\{x \in \Omega_1; T(x) \in E\}.$$

(a) Show that μ_2 is a measure on $\mathcal{F}_2$.
(b) For $E \in \mathcal{F}_2$, let 1_E be the indicator functon of E, that is, $1_E(y) = 1$ if $y \in E$ and $1_E(y) = 0$ if $y \notin E$. Show that

$$\int_{\Omega_1} 1_E(T(x))d\mu_1(x) = \int_{\Omega_2} 1_E(y)d\mu_2(y).$$

(c) Show that

$$\int_{\Omega_1} f(T(x))d\mu_1(x) = \int_{\Omega_2} f(y)d\mu_2(y),$$

for all non-negative measurable functions $f : \Omega_2 \longrightarrow \mathbb{R}$.
(d) Show that f is integrable w.r.t. μ_2 if and only if $f(T)$ is integrable w.r.t. μ_1, and that in that case the formula in (c) still holds.

5. Let μ_1 and ν_1 be finite measures on $(\Omega_1, \mathcal{F}_1)$, and μ_2 and ν_2 finite measures on $(\Omega_2, \mathcal{F}_2)$. Furthermore, suppose that for all $A_1 \in \mathcal{F}_1$ we have $\nu_1(A_1) = \int_{A_1} f d\mu_1$, and for all $A_2 \in \mathcal{F}_2$ we have $\nu_2(A_2) = \int_{A_2} g d\mu_2$. Note that f and g are Radon-Nikodym derivatives. Consider $\nu = \nu_1 \times \nu_2$ and $\mu = \mu_1 \times \mu_2$ on $\Omega = \Omega_1 \times \Omega_2$. Let $h(x, y) = f(x)g(y)$; h is a map from $\Omega \longrightarrow \mathbb{R}$.
(a) Show, using Fubini's theorem, that

$$\nu(A_1 \times A_2) = \int_{A_1 \times A_2} h d\mu.$$

(b) Show that

$$\mathcal{H} = \{A \in \mathcal{F}_1 \times \mathcal{F}_2; \nu(A) = \int_A h d\mu\}$$

is a monotone class.
(c) Formulate the monotone class theorem.
(d) Use the monotone class theorem to show that for all $A \in \mathcal{F}_1 \times \mathcal{F}_2$ we have

$$\nu(A) = \int_A h d\mu.$$