Solutions

1. (a) Write D_n for all data. Then

$$L(\theta \mid D_n) = \prod_{i=1}^n f_\theta(X_i, Y_i) = \exp\left(\theta S_X - \frac{1}{\theta} S_Y\right).$$

Here, $S_X = \sum_{i=1}^n X_i$ and $S_Y = \sum_{i=1}^n Y_i$. The loglikelihood equals

$$\ell(\theta; D_n) = \theta S_X - \frac{1}{\theta} S_Y.$$

Differentiating and equating to zero gives

$$\dot\ell(\theta; D_n) = -S_X + \frac{1}{\theta^2} S_Y = 0$$

which implies $\theta^2 = S_Y/S_X$. We need to have $\theta > 0$, the loglikelihood is concave, thus $\hat\Theta = \sqrt{S_Y/S_X}$.

(b) From part (a) the factorisation theorem immediately implies that (S_X, S_Y) is sufficient for θ.

(c)

$$I(\theta) = -E\ell''(\theta \mid D_1) = \frac{2}{\theta^3} EY_1 = \frac{2}{\theta^3} \int\int y e^{-\theta x - y/\theta} dx dy = \frac{2}{\theta^2}.$$

(d) 0 and $1/I(\theta) = \theta^2/2$.

2. (a)

$$L(\theta) = \prod_{i=1}^n \frac{1}{\theta} \mathbf{1}_{[0,\theta]}(x_i) = \theta^{-n} \mathbf{1}_{[x_{(n)},\infty)}(\theta).$$

$\pi(\theta \mid x) \propto L(\theta)\pi(\theta) \propto \theta^{-n}\mathbf{1}_{[x_{(n)},\infty)}(\theta)\cdot\theta^{-(\alpha+1)}\mathbf{1}_{[\beta,\infty)}(\theta) = \theta^{-(\alpha+n+1)}\mathbf{1}_{\max([x_{(n)},\beta),\infty)}(\theta)$.

This gives $\alpha_p = \alpha + n$, $\beta_p = \max([x_{(n)}, \beta)$.

(b) We need to solve for $\int_{\beta_p}^c \pi(\theta \mid x) d\theta = 0.95$. That is, with slight abuse of notation, $P(\mathrm{Par}(\alpha_p, \beta_p) \le c) = 0.95$. Thus $1 - (\beta_p/c)^{\alpha_p} = 0.95$, from which it follows that $(\beta_p/c)^{\alpha_p} = 0.05$.

(c) We choose H_1 if

$$\int_k^\infty \pi(\theta \mid x) d\theta > \int_0^k \pi(\theta \mid x) d\theta.$$

Equivalently, choose H_1 if $\int_k^\infty \pi(\theta \mid x) d\theta > 1/2$. Thus

$$\left(\frac{\beta_p}{k}\right)^{\alpha_p} > 1/2.$$

As $X_{(n)} > \beta$, we have $\beta_p = X_{(n)}$ from which the claim follows.

3. (a) The NP-test is most powerful. The test statistic is given by

$$\Lambda = \frac{L(2)}{L(1)} = \prod_{i=1}^{n} \frac{B(1,1)}{B(2,2)} X_i(1-X_i)$$
$$= \left(\frac{B(1,1)}{B(2,2)}\right)^n \prod_{i=1}^{n} X_i(1-X_i)$$

Hence, we reject for large value of

$$\log \prod_{i=1}^{n} X_i(1-X_i) = \sum_{i=1}^{n} \left(\log X_i + \log(1-X_i)\right).$$

(b) i.

$$\mathbb{P}_{\theta=1}(1/2 - c \leq X_1 \leq 1/2 + c) = 0.05.$$

Under $\theta = 1$, $X_1 \sim Unif(0,1)$. Therefore, this leads to $2c = 0.05$ which gives $c = 0.025$.

ii. The power is given by (simple alternative)

$$\mathbb{P}_{\theta=2}(1/2 - c \leq X_1 \leq 1/2 + c) = \frac{\int_{1/2-c}^{1/2+c} x(1-x)\mathbf{1}_{[0,1]}(x)dx}{\int_0^1 x(1-x)dx}$$

A primitive of the integrand is given by G. Hence, the numerator equals $G(c+1/2) - G(c-1/2)$. The denominator equals $1/6$. Combined, this gives the stated result.

4. (a) $Z_i = X_1/\sigma \sim N(0,1)$. This implies that $\sigma^{-2}\sum_i X_i^2 = \sum_i Z_i^2$. The latter distribution does not depend on σ.

(b) $\sum_i Z_i^2 \sim \chi^2_n$. Hence, with $U = \sum_i X_i^2$

$$P\left(\chi^2_{n,\alpha/2} \leq \sigma^{-2}U \leq \chi^2_{n,1-\alpha/2}\right) = 1 - \alpha.$$

Equivalently

$$P\left(\sqrt{\frac{U}{\chi^2_{n,1-\alpha/2}}} \leq \sigma \leq \sqrt{\frac{U}{\chi^2_{n,\alpha/2}}}\right) = 1 - \alpha.$$