Vrije Universiteit Amsterdam
Department of Mathematics

Exam Mathematical Statistics (XB_0049)
December 19, 2023, 12.15–14.30

You are not allowed to use calculators, phones, laptops or other tools. Clearly put your name and student number on all sheets that you submit. Your answer may contain quantiles from a distribution. Expressions for densities, expectation, variance and quantiles for some distributions are given on page 3 of the exam. You can use these expressions if needed for answering the questions below.

Unless stated differently, always add an explanation to your answer.

1. Consider a population with three kinds of individuals, say 1, 2 and 3 occurring in the following proportions: $p_1 = \theta^2$, $p_2 = 2\theta(1-\theta)$, $p_3 = (1-\theta)^2$ $(0 < \theta < 1)$. In a random sample of size n chosen from the population, n_1 individuals have been labeled 1, n_2 labelled 2, n_3 labelled 3, respectively $(n_1 + n_2 + n_3 = n)$.

   (a) [1 PT]. Write down the likelihood function for θ.

   (b) [3 PT]. Suppose we take the Bayesian point of view and use the Beta(2, 4)-distribution as a prior for θ. Show that if we observe $(n_1, n_2, n_3) = (3, 6, 4)$, then the posterior mean equals 7/16.

2. We say that the random variable X has the Shifted Geometric distribution with parameter $\theta \in (0, 1)$, denoted by $X \sim \mathrm{SGeo}(\theta)$, if it has probability mass function

$$f_X(x; \theta) = \mathrm{P}_\theta(X = x) = (1-\theta)^x \theta, \qquad x = 0, 1, \ldots.$$

   We have $\mathrm{E}_\theta X = (1-\theta)/\theta$. Suppose $X_1, \ldots, X_n$ are independent and identically distributed with $X_i \sim \mathrm{SGeo}(\theta)$.

   (a) [2 PT]. Derive a method-of-moments estimator for θ.

   (b) [2 PT]. Derive the maximum likelihood estimator for θ.

   (c) [1 PT]. What is the maximum likelihood estimator for $\tau = \theta/(1-\theta)$?

   (d) [1 PT]. Is the SGeo-distribution within the exponential family? If so, show it.

   (e) [2 PT]. Suppose we wish to test the hypothesis $H_0 : \theta = 1/4$ versus $H_1 : \theta = \theta_1$ with $\theta_1 > 1/4$. Derive the test-statistic which is optimal according to the Neyman-Pearson lemma.

   (f) [1 PT]. Show that the Neyman-Pearson test is equivalent to rejecting H_0 for small values of $\sum_{i=1}^n X_i$.

(g) [2 PT]. Now suppose $n = 1$ and that we obtain the realisation $x_1 = 2$. Compute the p-value of the test with $T := X_1$ as test statistic.

3. [3 PT]. The Cramer-Rao theorem is as follows: Suppose X has probability density function $x \mapsto f(x;\theta)$.

   Assume $\theta \in \mathbb{R}$ and let $\varphi(X)$ be a one-dimensional statistic with $\mathbb{E}_\theta|\varphi(X)| < \infty$ for all θ. Suppose "regularity conditions" are satisfied, which ensure $I(\theta) > 0$ and that $\int \varphi(x)f(x;\theta)dx$ can be differentiated under the integral sign with respect to θ. Then

$$\mathrm{Var}_\theta\, \varphi(X) \geq \frac{\left(\frac{d}{d\theta}\, \mathrm{E}_\theta\, \varphi(X)\right)^2}{I(\theta)},$$

   where $I(\theta)$ denotes the Fisher-information.

   The first part of the proof of this theorem consists of showing

$$\frac{d}{d\theta}\, \mathrm{E}_\theta\, \varphi(X) = \mathbb{E}_\theta\left[\varphi(X)s(\theta;X)\right], \tag{1}$$

   where $s(\theta;x) = \frac{d}{d\theta}\log f(x;\theta)$. Give a proof of Equation (1).

4. Suppose that $X_1,\ldots,X_n$ are independent, where each X_i has the $N(\theta,\sigma^2)$ distribution. Here σ is assumed to be known.

   (a) [2 PT]. Derive an expression for the $(1-\alpha)100\%$-confidence interval for θ ($\alpha \in (0,1)$. See the page "Distributions and Notation" at the end of the exam for notation on lower- and upper quantiles for the standard normal distribution.

   (b) [2 PT]. Alternatively, we can follow the Bayesian approach where θ is viewed as a realization of the random variable Θ. Assume an improper prior $f_\Theta(\theta) = c$ ($c \in \mathbb{R}$) on θ (in the book this is also written as $\pi(\theta) = c$). Show that the posterior density is proportional to

$$\exp\left(-\frac{1}{2\sigma^2}n\theta^2 + \frac{\theta}{\sigma^2}n\bar{x}_n\right).$$

   (c) [2 PT]. From this expression we can derive that the posterior of distribution is the $N(\bar{x}_n, \sigma^2/n)$-distribution (you don't need to derive this and you can consider this to be given). With this result, derive a $(1-\alpha)100\%$-credible interval for θ.

Distributions and notation

- If $X \sim Unif\,(0, \theta)$, then $\mathbb{E}X = \theta/2$ and $\mathrm{Var}(X) = \theta^2/12$.
- If $X \sim N(\mu, \sigma^2)$, then its density is given by $p(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{(x-\mu)^2)}{2\sigma^2}\right)$ and in case $\mu - 0$ and $\sigma = 1$ we define $\Phi(x) := \mathbb{P}(X \leq x)$. The lower quantile ξ_α for the standard Normal distribution is defined by $\Phi(\xi_\alpha) = \alpha$, for $\alpha \in (0, 1)$. The upper quantile z_α for the standard Normal distribution is defined by $\Phi(z_\alpha) = 1 - \alpha$.
- If $X \sim \chi^2_k$, then $\chi^2_{k,\alpha}$ is defined by $\mathbb{P}(X \leq \chi^2_{k,\alpha}) = \alpha$.
- If $X \sim t_k$, then $t_{k,\alpha}$ is defined by $\mathbb{P}(X \leq t_{k,\alpha}) = \alpha$.
- If $X \sim Ga(\alpha, \beta)$, then its density is given by $p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbf{1}_{(0,\infty)}(x)$. $\mathbb{E}X = \alpha/\beta$, $\mathrm{Var}(X) = \alpha/\beta^2$. The special case of $\alpha = 1$ corresponds to the exponential distribution with parameter β.
- If $X \sim Bin(n, p)$, then $\mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, for $x \in \{0, 1, \ldots, n\}$. $\mathbb{E}X = np$, $\mathrm{Var}(X) = np(1-p)$.
- If $X \sim Beta(\alpha, \beta)$, then its density is given by $p(x) = B(\alpha, \beta)^{-1} x^{\alpha-1} (1-x)^{\beta-1} \mathbf{1}_{[0,1]}(x)$. $\mathbb{E}X = \alpha/(\alpha+\beta)$, $\mathrm{Var}(X) = \alpha\beta/\left((\alpha+\beta)^2(\alpha+\beta+1)\right)$