Solutions

1. (a) $L(\theta) = (\theta^2)^{n_1}(2\theta(1-\theta))^{n_2}((1-\theta)^2)^{n_3}$.

   (b) $L(\theta) \propto \theta^{2n_1+n_2}(1-\theta)^{n_2+2n_3}$. Hence, with $X = (N_1, N_2, N_3)$,

$$
\begin{aligned}
f_{\Theta|X}(\theta \mid x) &\propto \theta^{2n_1+n_2}(1-\theta)^{n_2+2n_3}\theta^{2-1}(1-\theta)^{4-1} \\
&= \theta^{2n_1+n_2+2-1}(1-\theta)^{n_2+2n_3+4-1}
\end{aligned}
$$

   Hence the posterior is $Beta(2n_1+n_2+2, n_2+2n_3+4)$. Plugging in $(n_1,n_2,n_3) = (3,6,4)$ we get $Beta(14,18)$. Therefore, the posterior mean equals $14/32 = 7/16$.

2. (a) MM-estimator T is defined by $\bar{X}_n = \frac{1-T}{T}$. Solving gives $T = 1/(\bar{X}_n + 1)$.

   (b) We have

$$\ell(\theta) = S\log(1-\theta) + n\log\theta,$$

   where $S = \sum_{i=1}^n X_i$.

$$\ell'(\theta) = \frac{n(1-\theta) - \theta S}{\theta(1-\theta)}.$$

   Setting $\ell'(\theta) = 0$ gives $\theta = \frac{n}{n+S}$. It's easily verified that this stationary point indeed corresponds to a maximum.

   (c) As $g\colon (0,1) \to \mathbb{R}$ with $g(\theta) = \theta/(1-\theta)$ is bijective, the MLE for τ is given by

$$g\left(\frac{n}{n+S}\right) = \frac{n}{S}.$$

   (d) Yes, since it can be written as

$$f_X(x;\theta) = \exp(x\log(1-\theta) + \log\theta).$$

   (e) The optimal test is the Neyman-Peason test which rejects for large values of

$$\frac{L(\theta_1;X)}{L(1/4;X)} = \frac{\prod_{i=1}^n (1-\theta_1)^{X_i}\theta_1}{\prod_{i=1}^n (1-1/4)^{X_i}1/4} = \left(\frac{1-\theta_1}{1-1/4}\right)^{\sum_{i=1}^n X_i}(4\theta_1)^n.$$

   (f) As $\theta_1 > 1/4$, we have $(1-\theta_1)/(1-1/4) < 1$ and we reject for small values of $\sum_{i=1}^n X_i$.

   (g) The p-value is given by

$$
\begin{aligned}
p &= \mathrm{P}_{1/4}(\mathrm{X} \le 2) = \mathrm{P}_{1/4}(\mathrm{X}=0) + \mathrm{P}_{1/4}(\mathrm{X}=1) + \mathrm{P}_{1/4}(\mathrm{X}=2) \\
&= \frac{1}{4}\left(1 + (3/4) + (3/4)^2\right) = \frac{37}{64}.
\end{aligned}
$$

3.

$$\begin{aligned}\frac{d}{d\theta}\operatorname{E}_\theta\varphi(X) &= \frac{d}{d\theta}\int \varphi(x) f_X(x;\theta)dx \\ &= \int \varphi(x)\frac{d}{d\theta}f_X(x;\theta)dx \\ &= \int \varphi(x)s(\theta;x)f_X(x;\theta)dx = \mathbb{E}_\theta\left[\varphi(X)s(\theta;X)\right],\end{aligned}$$

4. (a) $\sqrt{n}\frac{\bar{X}_n-\theta}{\sigma} \sim N(0,1)$ Therefore

$$P\left(-z_{\alpha/2} \leq \sqrt{n}\frac{\bar{X}_n-\theta}{\sigma} \leq z_{\alpha/2}\right) = 1-\alpha.$$

Rewriting gives

$$\left[\bar{x}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right].$$

(b) The posterior density satisfies

$$\pi(\theta \mid x) \propto \prod_{i=1}^{n}\exp\left(-\frac{1}{2\sigma^2}(X_i-\theta)^2\right) = \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\theta)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}n\theta^2 + \frac{\theta}{\sigma^2}n\bar{x}_n\right)$$

(c)

$$P\left(-z_{\alpha/2} \leq \sqrt{n}\frac{\Theta-\bar{x}_n}{\sigma} \leq z_{\alpha/2} \mid X = x\right) = 1-\alpha.$$

Rewriting gives that the credible interval is given by

$$\left[\bar{x}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right].$$