Vrije Universiteit Amsterdam
Department of Mathematics

Midterm Mathematical Statistics (XB_0049)
October 24, 2022, 12.15–14.30

You are not allowed to use calculators, phones, laptops or other tools. Clearly put your name and student number on all sheets that you submit.
Unless stated differently, always add an explanation to your answer.

1. For $-\infty < s < t < \infty$, the distribution function of the uniform distribution on the interval $[s, t]$, is given by

$$F(x) = \begin{cases} 0 & \text{if } x < s \\ \dfrac{x-s}{t-s} & \text{if } s \leq x \leq t \; . \\ 1 & \text{if } x > t \end{cases}$$

   (a) [2 pt]. Does the uniform distribution with parameters s and t belong to a location-scale family $\{F_{a,b} : a \in \mathbb{R}, b > 0\}$ associated with the standard uniform distribution on $[0, 1]$? *Hint: note that $F_{a,b}$ is the distribution function of the random variable $a + bX$, where $X \sim Unif\,(0, 1)$.*
   If yes, then specify the location parameter a and scale parameter b. If no, then explain why.

   (b) [2 pt]. Derive the expression of the quantile function corresponding to the uniform distribution on the interval $[s, t]$.

2. Let $X_1, \ldots, X_n$ be independent and identically distributed. Assume each X_i has the geometric distribution with parameter θ,

$$P_\theta(X_1 = k) = (1 - \theta)^{k-1}\theta, \qquad k = 1, 2, \ldots\,,$$

   where $0 \leq \theta \leq 1$ is unknown.

   (a) [3 pt]. Compute the maximum likelihood estimator for θ.

   (b) [3 pt]. As prior distribution for θ, we choose probability density

$$\pi(\theta) = 6(1 - \theta)\theta\mathbf{1}_{(0,1)}(\theta).$$

   Compute the posterior density and report to which distribution it corresponds.
   Hint: if $Z \sim Be\,(\alpha, \beta)$ (meaning Z is distributed according to the Beta-distribution with parameters α and β), then $p(z) \propto z^{\alpha-1}(1 - z)^{\beta-1}\mathbf{1}_{[0,1]}(z)$.

3. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed according to the Poisson(θ)-distribution. That is, for $\theta > 0$

$$p_\theta(x) = e^{-\theta}\frac{\theta^x}{x!}, \qquad x = 0, 1, \ldots.$$

Define $S_n = \sum_{i=1}^n X_i$.

   (a) [2 pt]. Show that S_n is sufficient for θ.

   (b) [1 pt]. Suppose the estimator T is unbiased for θ. How can we construct from T another unbiased estimator for θ with smaller variance?

   (c) [2 pt]. Compute $\mathrm{E}_\theta[X_1 \mid S_n]$.
   Hint: note that $\mathrm{E}_\theta[X_i \mid S_n]$ *is the same for all* $1 \leq i \leq n$ *and that* $\sum_{i=1}^n \mathrm{E}_\theta[X_i \mid S_n] = S_n$.

   (d) [1 pt]. Explain why the result under (d) does not depend on θ.

4. Let $X_1, \ldots, X_n$ be independent and identically distributed with the $N(\mu, 1)$ distribution. We test $H_0 : \mu \leq 0$ against $H_1 : \mu > 0$ with test statistic $T = \overline{X}$. The critical region for T is given by $K = [c, \infty)$, for some $c > 0$. Let Φ denote the cumulative distribution function of the standard normal distribution.

   (a) [2 pt]. Give the definition of the size of this test and determine an expression for it in terms of c and Φ.

   (b) [2 pt]. Give the definition of the power function of this test and determine an expression for it in terms of c and Φ.

5. [2 pt]. Consider a statistical model with observation X distributed according to distribution P_θ. Suppose that T is the Bayes estimator for θ. Assume $\mathrm{MSE}(\theta; T)$ is constant (it does not depend on θ). Show that T is minimax.