Department of Mathematics	Midterm exam Mathematical Statistics
VU	25 October 2021

- **You may not use calculators, phones, laptops, or other tools.**
- **Clearly put your name and student id on all sheets you hand in.**

Good luck!

1. Let $X_1, \ldots, X_n$ be i.i.d. random variables with density p_θ given by

$$p_\theta(x) = \frac{1}{2\theta} e^{-\frac{|x|}{\theta}}, \qquad x \in \mathbb{R},$$

   where $\theta > 0$ is an unknown parameter.

   (a) Compute the first two moments $\mathbb{E}_\theta X_1$ and $\mathbb{E}_\theta X_1^2$ of X_1 and determine the method of moments estimator for θ.

   (b) Determine the maximum likelihood estimator for θ.

   (c) Determine a sufficient and complete statistic for the parameter θ.

   (d) Determine an UMVU estimator for θ.

2. Let $X_1, \ldots, X_n$ be independent and Poisson distributed with parameter $\lambda > 0$. Hence the density of X_i is given by

$$p_\lambda(k) = \mathbb{P}_\lambda(X_i = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots.$$

   We want to make Bayesian inference about the parameter λ. As prior distribution we choose a gamma distribution, with parameters $\alpha, \beta > 0$. This is the distribution with density

$$\pi(\lambda) = \frac{\lambda^{\alpha-1}\beta^\alpha e^{-\beta\lambda}}{\Gamma(\alpha)}, \qquad \lambda > 0,$$

   where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

   is Euler's gamma function.

   (a) Determine the posterior distribution of λ.

(b) Determine the Bayes estimator for λ. You may use the fact that the expectation and variance of a gamma distribution with parameters α and β are given by α/β and α/β^2, respectively.

(c) Show that for $n \to \infty$, the variance of the posterior distribution behaves like the Cramér-Rao lower bound for the variance of an unbiased estimator for λ (in correspondence with the so-called Bernstein-von Mises theorem).

3. Let $X_1, \ldots, X_n$ be independent and uniformly distributed on the interval $[0, \theta]$, where $\theta > 0$ is an unknown parameter.

   (a) Determine the maximum likelihood estimator $\hat{\theta}_n$ for θ.

   (b) Show that for every $\theta > 0$, the distribution function $x \mapsto \mathbb{P}_\theta(n(\theta - \hat{\theta}_n) \leq x)$ converges pointwise on $(0, \infty)$ to the distribution function of an exponential distribution with parameter $1/\theta$, which is given by $x \mapsto 1 - e^{-x/\theta}$.

   (c) Is this estimation problem asymptotically (for large n) harder or easier than a "regular" problem in which the regularity conditions are fulfilled that imply the usual asymptotic normality of the MLE? Motivate your answer.

4. Maurice is interested in the fraction p of all Dutch voters who are in favour of having Mark Rutte as prime minister again. He calls 100 people at random and asks them whether they are in favour of this. Let X be the number of people that answer to the affirmative.

   (a) What is a reasonable statistical model for this problem?

   (b) Maurice wants to know if the fraction p is greater than 1/2. Formulate this as a testing problem (i.e. give the hypotheses).

   (c) Give the derivation of the standard test of level $\alpha = 0.05$ for this problem. You may use the fact that for the distribution function F of the binomial distribution with parameters $n = 100$ and $p = 1/2$ it holds that

   $$F(55) = 0.86, \quad F(56) = 0.90, \quad F(57) = 0.93, \quad F(58) = 0.96, \quad F(59) = 0.97.$$

   (d) Suppose that 56 people answer to the affirmative. Can Maurice then conclude at uncertainty level 5% that indeed $p > 1/2$?

Norming:

1(a): 3	**2(a): 3**	**3(a): 2**	**4(a): 1**
1(b): 3	**2(b): 2**	**3(b): 3**	**4(b): 1**
1(c): 3	**2(c): 2**	**3(c): 2**	**4(c): 3**
1(d): 2			**4(d): 1**

Grade = max{1, 10*fraction of points obtained}.