1. (a) Since p_θ is symmetric around 0, we have $E_\theta X_1 = 0$ and

$$E_\theta X_1^2 = \int_0^\infty \frac{1}{\theta} x^2 e^{-\frac{x}{\theta}} dx = -\int_0^\infty x^2 \, d e^{-\frac{x}{\theta}}$$

$$= 2\int_0^\infty e^{-\frac{x}{\theta}} x \, dx = -2\theta \int_0^\infty x \, d e^{-\frac{x}{\theta}}$$

$$= 2\theta^2 \int_0^\infty \frac{1}{\theta} e^{-\frac{x}{\theta}} dx = 2\theta^2.$$

(Also Ok if you know the second moment of an exponential).

For the moment estimator, can not use first moment, so we use the second. The est. $\hat{\theta}_{MM}$ is then solution of

$2\theta^2 = \overline{X^2}$. Hence $\hat{\theta}_{MM} = \sqrt{\frac{1}{2}\overline{X^2}}$.

(b) • Likelihood: $L(\theta) = \left(\frac{1}{2}\right)^n \cdot \theta^{-n} e^{-\frac{1}{\theta}\sum|X_i|}$

• log ": $l(\theta) = \log L = n \log 1/2 - n \log \theta - \frac{1}{\theta}\sum|X_i|$

• score: $\dot{l}(\theta) = \frac{\partial}{\partial\theta} l = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum|X_i| = \frac{\sum|X_i| - n\theta}{\theta^2}$

• setting eq. to 0 and checking gives $\hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^n |X_i|$

(c) We have an exponential family with sufficient statistic $\sum|X_i|$ and $Q(\theta) = -\frac{1}{\theta}$. Since $\{-\frac{1}{\theta} : \theta > 0\} = (-\infty, 0)$ has an interior point, $\sum|X_i|$ is complete as well

(d) The MLE is based on a sufficient and complete statistic. Moreover,

$$\mathbb{E}_\theta \hat{\theta}_{MLE} = \mathbb{E}_\theta |X_1| = \int_0^\infty \frac{1}{\theta} x e^{-x/\theta} dx = \theta,$$

hence the MLE is also unbiased, hence UMVU.

2. (a) Likelihood: $L(\lambda) = \left(\prod \frac{1}{X_i!}\right) \lambda^{\sum X_i} e^{-n\lambda}$

Hence posterior density:

$$p_{\Lambda|X_1,\dots,X_n}(\lambda) \propto \pi(\lambda) L(\lambda)$$
$$\propto \lambda^{\sum X_i + \alpha - 1} \cdot e^{-(n+\beta)\lambda}.$$

So the posterior is gamma with parameters $\sum X_i + \alpha$ and $n+\beta$

(b) The Bayes estimator is the expectation of the posterior. By (a), this is

$$\hat{\theta}_{Bayes} = \frac{\sum X_i + \alpha}{n+\beta}$$

(c) By (a), the variance of the posterior is $\dfrac{\sum X_i + \alpha}{(n+\beta)^2}$. Since, by the LLN,

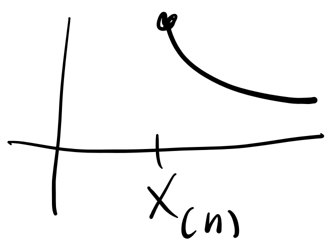
$\frac{1}{n}\sum X_i \xrightarrow{P} \mathbb{E}_\lambda X_1 = \lambda$, this behave like λ/n for large n.

The Fisher information in a single observation is

$$i_\lambda = \mathrm{Var}_\lambda \frac{\partial}{\partial \lambda}\log p_\lambda(X_1) = \mathrm{Var}_\lambda\left(\frac{X_1}{\lambda}-1\right) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.$$

Hence, the CR lower bound is also $\frac{1}{n i_\lambda} = \frac{\lambda}{n}$.

3. (a) Likelihood: $L(\theta) = \frac{1}{\theta^n} 1_{X_{(n)} \le \theta}$.



From the picture $\hat{\theta}_n = X_{(n)}$.

(b) For $x > 0$ and large n,

$$\mathbb{P}_\theta\left(n(\theta - X_{(n)}) > x\right) = \mathbb{P}_\theta\left(X_{(n)} < \theta - \frac{x}{n}\right)$$

$$= \mathbb{P}_\theta\left(X_1 < \theta - \frac{x}{n}; \dots; X_n < \theta - \frac{x}{n}\right) = \left(\mathbb{P}_\theta\left(X_1 < \theta - \frac{x}{n}\right)\right)^n$$

$$= \left(\frac{\theta - \frac{x}{n}}{\theta}\right)^n = \left(1 - \frac{x}{n\theta}\right)^n \longrightarrow e^{-\frac{x}{\theta}}.$$

(c) We have that $\hat{\theta}_n$ converges to θ at the rate $\frac{1}{n}$. This is much faster than the "regular" rate $\frac{1}{\sqrt{n}}$. Hence, the problem is easier.

4. (a) $X \sim \text{Bin}(100, p)$

(b) $H_0 : p \leq 1/2$; $H_1 : p > 1/2$

(c) • Large values of X indicate that H_1 is true. So test of the form "reject H_0 if $X \geq c$" for some $c \in \{0, \dots, 100\}$.

• Want a test of level $\alpha = 0.05$, i.e.

$$\sup_{p \leq 1/2} \underbrace{\mathbb{P}_p(X \geq c)}_{\uparrow \text{ in } p} \leq 0.05$$

$$\Leftrightarrow \mathbb{P}_{1/2}(X \geq c) \leq 0.05$$

$$\Leftrightarrow \mathbb{P}_{1/2}(X \leq c-1) \geq 0.95 \Leftrightarrow F(c-1) \geq 0.95$$

• We want c as small as possible, so that $c - 1 = 58 \Leftrightarrow c = 59$.

(d) For $X = 56$ H_0 can <u>not</u> be rejected, so M. can not conclude that.