Mathematical Optimization – Exam February 2022

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The exam consists of five exercises worth a total of 100 *points* $= 12+15+25+23+25$. *The exam lasts for 2h and 45min. Do not forget to add explanations/comments next to your algebraic/numerical solutions, especially where requested. You are allowed to use a two-sided A4 hand-written cheat sheet that needs to be handed-in together with your exam. It is highly recommended that you write your resolution using a pen and not a pencil.*

Exercise 1 [12 points = 2 + 5 + 5]

Consider the following linear program:

$$\begin{aligned} \max \quad & 5x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 9 \\ & x_1 \geq 0 \\ & x_2 \geq 1. \end{aligned}$$

(a) [2 points] Rewrite the constraints in the form $A\boldsymbol{x} \geq \boldsymbol{b}$. Is A the totally unimodular matrix? What can we deduce about the integrality of the optimal solution of the problem?

(b) [5 points] Represent the problem graphically, including the lines defining the constraints, the resulting feasible region, and the isolines of the objective function.

(c) [5 points] Using the graphical representation in (a), solve the problem by report the optimal solution and the optimal value achieved.

Exercise 2 [15 points = 5 + 5 + 5]

Consider the following linear program:

$$\begin{aligned} \max \quad & x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_4 \geq 5 \\ & 6x_1 + x_4 \leq 10 \\ & 2x_1 + x_2 + 5x_3 = 9 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \\ & x_3 \in \mathbb{R} \\ & x_4 \geq 0. \end{aligned}$$

(a) [5 points] Write the dual problem of the above linear program.

(b) [5 points] Consider the point $\boldsymbol{x} \in \mathbb{R}^4$ defined by $x_1 = x_2 = 0, x_3 = 1.8, x_4 = 10$. Is $\boldsymbol{x}$ a feasible point for the primal problem? Motivate your answer. Based on this, what can you say about the optimal solution value of the dual problem?

(c) [5 points] Is the point $\boldsymbol{x} \in \mathbb{R}^4$ given in (b), that is $x_1 = x_2 = 0,\ x_3 = 1.8,\ x_4 = 10$, also an optimal solution for the primal problem? Motivate your answer. Based on this, what can you conclude about the optimal solution of the dual problem?

Exercise 3 [25 points = 5 + 5 + 5 + 5 + 5]

Consider the following optimization problem:

$$\begin{aligned} \min \quad & x_1^4 + x_2^2 + x_3^4 \\ \text{s.t.} \quad & x_1^2 + 2x_2^2 \leq 10 \\ & 5 - x_1 - x_3^4 \geq 0 \\ & x_1 + x_2 = 2 \\ & x_1, x_2, x_3 \in \mathbb{R}. \end{aligned}$$

(a) [5 points] Is the optimization problem convex? Motivate your answer.

(b) [5 points] After having rewritten the problem in standard form, write the corresponding Lagrangian function.

(c) [5 points] Write the Karush–Kuhn–Tucker (KKT) conditions for this problem.

(d) [5 points] If you were to solve the KKT conditions, will you obtain the optimal solution of the problem? Motivate your answer. *Note that you are not asked to solve the system of KKT conditions.*

(e) [5 points] Consider the first two constraints, namely $x_1^2 + 2x_2^2 \leq 10$ and $x_1 + x_3^4 \leq 5$. Can each of them (separately) be represented as a Second-Order Conic constraint ? If yes, show how. If not, motivate your answer.

Exercise 4 [23 points = 10 + 7 + 6]

Consider the following optimization problem:

$$\begin{aligned} \min \quad & \max\{3x_1, x_1 + 6x_2\} \\ \text{s.t.} \quad & 2x_1x_2 + x_1 \geq 1 \\ & x_1 - 3x_2 \geq -2 \\ & x_1 \leq 10 \\ & x_1 \geq 0 \\ & x_2 \in \{0, 1\}. \end{aligned}$$

(a) [10 points] Reformulate the problem as a (integer) linear program, modifying both the objective and the nonlinear constraint.

Assume now that the coefficients on the left-hand side of the second constraint, i.e., $x_1 - 3x_2 \geq -2$, are now uncertain and thus we replace it with the following chance constraint

$$\mathbb{P}(z_1x_1 + z_2x_2 \geq -2) \leq 1 - \varepsilon,$$

where $\varepsilon \in (0, 1)$ and $\boldsymbol{z} = (z_1, z_2)$ is the vector of uncertain coefficients.

(b) [7 points] Assuming the vector of uncertain coefficients $\boldsymbol{z} = (z_1, z_2)$ follows a multivariate Gaussian distribution $(\boldsymbol{\mu}, \Sigma)$ with mean $\boldsymbol{\mu} = (1, -3)$ and covariance matrix $\Sigma = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, rewrite the chance constraint above in an equivalent deterministic form.

(c) [6 points] Under the same assumptions as in (b), is the chance constraint introduced above convex for all values of ε? Motivate your answer in either case.

Exercise 5 [25 points = 10 + 10 + 5]

Taif is a company that manufactures cars in two factories and then ships them to three regions in Europe. The two factories, labelled as A and B, can supply at most 450 and 600 cars, respectively. The customer demands in region 1, 2, and 3 are equal to 450, 200 and 300 cars, respectively. To ship a car from each factory to each region Taif must pay a shipping cost (*unit shipping costs*). This cost is however subject to uncertainty, depending on different unpredictable factors (weather, traffic, trucks availability, etc.).

Taif has some data available regarding the *unit shipping costs* of previous shipping. After analyzing such data, Taif was able to derive a minimum and a maximum unit shipping cost from each factory and region pair, as reported in the following table (values in €/car).

	Region 1		Region 2		Region 3	
	min	max	min	max	min	max
Factory A	131	150	218	230	266	270
Factory B	250	280	116	120	263	270

Additionally, Taif observed that the sum of all the 6 *unit shipping costs* (from every possible factory to every possible region) never exceeds a total amount $C_{\max} = 1500$ €/car.

Taif wants to find the lowest-cost shipping plan for meeting the demands of the four regions without exceeding the capacities of the factories in the *worst-case*, taking into consideration the information on the minimum and maximum observed unit shipping cost for each connection and on the sum of all the unit shipping costs C.

(a) [10 points] Formulate a robust mathematical optimization model that Taif can use to solve its shipping cost minimization problem. Introduce and define the decision variables, describe the constraints, distinguishing the deterministic ones and those affected by uncertainty, and explicitly describe the uncertainty set in which the uncertain parameters vary. What type of uncertainty set is it?

(b) [10 points] Identify the adversarial problem of your robust model, define its dual, and derive the tractable robust counterpart of the problem.
Hint: First use one of the standard tricks to move the uncertainty from the objective to a constraint.

(c) [5 points] Assume that the maximum value of the total sum of the 6 *unit shipping costs* is decreased to $C_{\max} = 1400$ €/car. What is the relationship between the optimal solution cost of the new robust problem (obtained setting $C_{\max} = 1400$) and the solution of the previously formulated robust problem ($C_{\max} = 1500$)?