Mathematical Optimization – Exam February 2022
full resolution

dr. R. Paradiso dr. A. Zocca

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The exam consists of five exercises worth a total of 100 *points* $= 12+15+25+23+25$. *The exam lasts for 2h and 45min. Do not forget to add explanations/comments next to your algebraic/numerical solutions, especially where requested. You are allowed to use a two-sided A4 hand-written cheat sheet that needs to be handed-in together with your exam. It is highly recommended that you write your resolution using a pen and not a pencil.*

Exercise 1 [12 points = 2 + 5 + 5]

Consider the following linear program:

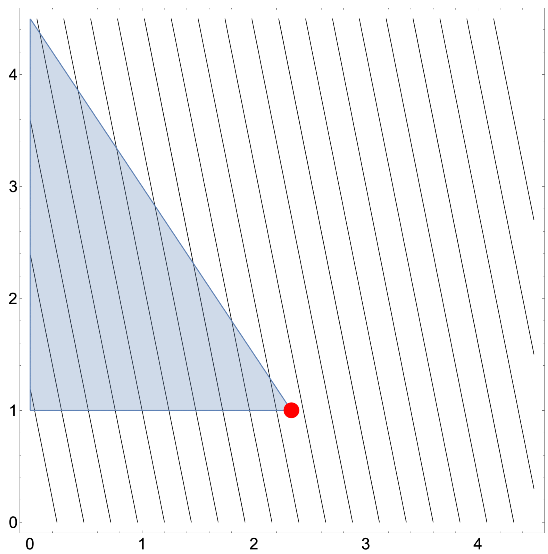
$$\begin{aligned} \max \quad & 5x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 9 \\ & x_1 \geq 0 \\ & x_2 \geq 1. \end{aligned}$$

(a) [2 points] Rewrite the constraints in the form $A\boldsymbol{x} \geq \boldsymbol{b}$. Is A the totally unimodular matrix? What can we deduce about the integrality of the optimal solution of the problem?

We can rewrite the constraint set as $A\boldsymbol{x} \geq \boldsymbol{b}$ by taking $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{b} = (-9, 0, 1)^\top$. The matrix A is not totally unimodular because it has at least one entry, i.e., A_{11}, that is not in the set $\{-1, 0, 1\}$. Thus, even if the vector $\boldsymbol{b} = (-9, 0, 1)$ is integer, there is no guarantee a priori that the solution will be integer.

(b) [5 points] Represent the problem graphically, including the lines defining the constraints, the resulting feasible region, and the isolines of the objective function.

See plot below, with the feasible region in blue and the isolines of the objective function in black.



(c) [5 points] Using the graphical representation in (a), solve the problem by reporting the optimal solution and the optimal value achieved.

The optimal solution is $x_1 = 7/3, x_2 = 1$ (displayed in red in the plot above) with a corresponding optimal value equal to $v(P) = 38/3$.

Exercise 2 [15 points = 5 + 5 + 5]

Consider the following linear program:

$$\begin{aligned} \max \quad & x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_4 \geq 5 \\ & 6x_1 + x_4 \leq 10 \\ & 2x_1 + x_2 + 5x_3 = 9 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \\ & x_3 \in \mathbb{R} \\ & x_4 \geq 0. \end{aligned}$$

(a) [5 points] Write the dual problem of the above linear program.

Using the linear duality theory, one can derive that the dual problem is

$$\begin{aligned} \min \quad & 5\lambda_1 + 10\lambda_2 + 9\lambda_3 \\ \text{s.t.} \quad & 3\lambda_1 + 6\lambda_2 + 2\lambda_3 \geq 1 \\ & 2\lambda_1 + \lambda_3 \leq 1 \\ & 5\lambda_3 = 3 \\ & \lambda_1 + \lambda_2 \geq 0 \\ & \lambda_1 \leq 0 \\ & \lambda_2 \geq 0 \\ & \lambda_3 \in \mathbb{R}. \end{aligned}$$

(b) [5 points] Consider the point $\boldsymbol{x} \in \mathbb{R}^4$ defined by $x_1 = x_2 = 0, x_3 = 1.8, x_4 = 10$. Is $\boldsymbol{x}$ a feasible point for the primal problem? Motivate your answer. Based on this, what can you say about the optimal solution value of the dual problem?

The given point is feasible because all constraints of the primal problem are satisfied. Since the primal problem is feasible, due to the Weak Duality Theorem, the dual problem can not be unbounded.

(c) [5 points] Is the point $\boldsymbol{x} \in \mathbb{R}^4$ given in (b), that is $x_1 = x_2 = 0,\ x_3 = 1.8,\ x_4 = 10$, also an optimal solution for the primal problem? Motivate your answer. Based on this, what can you conclude about the optimal solution of the dual problem?

One of the results seen at lecture for linear duality states that, since strong duality holds, complementary slackness must be satisfied. The given point can easily be shown to be feasible. We thus write the complementary slackness conditions

$$\begin{cases} \lambda_1(3x_1 + 2x_2 + x_4 - 5) = 0 & = & 0 \\ \lambda_2(6x_1 + x_4 - 10) & = & 0 \\ \lambda_3(2x_1 + x_2 + 5x_3 - 9) & = & 0 \\ (3\lambda_1 + 6\lambda_2 + 2\lambda_3)x_1 & = & 0 \\ (2\lambda_1 + \lambda_3)x_2 & = & 0 \\ (5\lambda_3 - 3)x_3 & = & 0 \\ (\lambda_1 + \lambda_2)x_4 & = & 0 \end{cases}$$

Plugging in the values $x_1 = x_2 = 0,\ x_3 = 1.8,\ x_4 = 10$ and solving the resulting system reduces to

$$\begin{cases} 5\lambda_1 & = & 0 \\ \lambda_3 & = & 3/5 \\ \lambda_1 & = & -\lambda_2 \end{cases}$$

and has an unique solution $\lambda_1 = \lambda_2 = 0,\ \lambda_3 = 0.6$. This solution is feasible for the dual problem and therefore optimal. Since the system has a solution, the point $x_1 = x_2 = 0, x_3 = 1.8, x_4 = 10$ is the optimal solution of the primal.

Exercise 3 [25 points = 5 + 5 + 5 + 5 + 5]

Consider the following optimization problem:

$$\begin{aligned} \min \quad & x_1^4 + x_2^2 + x_3^4 \\ \text{s.t.} \quad & x_1^2 + 2x_2^2 \leq 10 \\ & 5 - x_1 - x_3^4 \geq 0 \\ & x_1 + x_2 = 2 \\ & x_1, x_2, x_3 \in \mathbb{R}. \end{aligned}$$

(a) [5 points] Is the optimization problem convex? Motivate your answer.

The given minimization problem is convex because, after writing it in the standard form, its objective function and all its constraints are.

- The objective function is convex on its entire domain, being the sum of three convex functions. Alternatively, the gradient of the objective function $f(\boldsymbol{x})$ is $\nabla f(\boldsymbol{x}) = (4x_1^3, 2x_2, 4x_3^3)$ and the Hessian is

$$\nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} 12x_1^2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12x_3^2 \end{bmatrix}$$

which is a positive semidefinite matrix, having eigenvalues $12x_1^2$, 2 and $12x_3^2$, all nonnegative for all $\boldsymbol{x} \in \mathbb{R}^3$. This can be shown also by calculating the principal minors: the determinants of all the principal submatrices are bigger than or equal to 0 for every $\boldsymbol{x} \in \mathbb{R}^3$.

- The first nonlinear inequality constraint, once rewritten in standard form $g_1(\boldsymbol{x}) = x_1^2 + 2x_2^2 - 10 \leq 0$, is convex on its whole domain, being the sum of two convex functions. Alternatively, one can calculate the gradient as $\nabla g_1(\boldsymbol{x}) = (2x_1, 4x_2, 0)$ and the Hessian as

$$\nabla^2 g_1(\boldsymbol{x}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and conclude that the latter is a positive semidefinite matrix. Indeed, $\nabla^2 g_1(\boldsymbol{x})$ has eigenvalues 4, 2, and 0, all nonnegative, or, equivalently, the determinants of all the principal submatrices are bigger than or equal to 0.

- The second nonlinear inequality constraint, once rewritten in standard form $g_2(\boldsymbol{x}) = x_1 + x_3^4 - 5 \leq 0$, is convex on its whole domain, being the sum of two convex functions. Alternatively, one can calculate the gradient as $\nabla g_2(\boldsymbol{x}) = (1, 0, 4x_3^3)$ and the Hessian as

$$\nabla^2 g_2(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12x_3^2 \end{bmatrix}$$

and conclude that the latter is a positive semidefinite matrix. Indeed, $\nabla^2 g_2(\boldsymbol{x})$ has eigenvalues $12x_3^2 \geq 0$, 0, and 0, all nonnegative, or, equivalently, the determinants of all the principal submatrices are bigger than or equal to 0.

- The equality constraint $x_1 + x_2 = 2$ is linear, hence convex.

(b) [5 points] After having rewritten the problem in standard form, write the corresponding Lagrangian function.

The problem in standard form is

$$\begin{aligned} \min \quad & x_1^4 + x_2^2 + x_3^4 \\ \text{s.t.} \quad & x_1^2 + 2x_2^2 - 10 \leq 0 \\ & x_1 + x_3^4 - 5 \leq 0 \\ & x_1 + x_2 - 2 = 0 \\ & x_1, x_2, x_3 \in \mathbb{R}. \end{aligned}$$

Introducing two dual variables $\lambda_1, \lambda_2 \geq 0$ for the inequality constraint and one dual variable $\mu \in \mathbb{R}$ for the equality constraint, the Lagrangian function is

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2, \mu) = x_1^4 + x_2^2 + x_3^4 + \lambda_1(x_1^2 + 2x_2^2 - 10) + \lambda_2(x_1 + x_3^4 - 5) + \mu(x_1 + x_2 - 2).$$

(c) [5 points] Write the Karush–Kuhn–Tucker (KKT) conditions for this problem.

The KKT conditions are

$$
\begin{array}{lll}
4x_1^3 + 2\lambda_1 x_1 + \lambda_2 + \mu & = 0 & \text{(stationarity)} \\
2x_2 + 4\lambda_1 x_2 + \mu & = 0 & \text{(stationarity)} \\
4x_3^3 + 4\lambda_2 x_3^3 & = 0 & \text{(stationarity)} \\
x_1^2 + 2x_2^2 - 10 & \leq 0 & \text{(primal feasibility)} \\
x_1 + x_3^4 - 5 & \leq 0 & \text{(primal feasibility)} \\
x_1 + x_2 - 2 & = 0 & \text{(primal feasibility)} \\
\lambda_1 & \geq 0 & \text{(dual feasibility)} \\
\lambda_2 & \geq 0 & \text{(dual feasibility)} \\
\lambda_1(x_1^2 + 2x_2^2 - 10) & = 0 & \text{(complementary slackness)} \\
\lambda_2(x_1 + x_3^4 - 5) & = 0 & \text{(complementary slackness)}
\end{array}
$$

(d) [5 points] If you were to solve the KKT conditions, will you obtain the optimal solution of the problem? Motivate your answer. *Note that you are not asked to solve the system of KKT conditions.*

The problem is convex therefore if Slater's condition holds, the KKT conditions are *necessary and sufficient optimality conditions*, i.e., solving the KKT conditions will give the optimal solution of the problem. To check if Slater's condition holds, we need to find a feasible solution such that all the affine constraints of the problem are satisfied and all the nonaffine constraints are *strictly satisfied*. The solution $x_1 = 2$, $x_2 = x_3 = 0$ strictly satisfies the first two constraints (nonaffine constraints) and satisfies the affine constraint (last constraint). The existence of such a solution implies that Slater's condition holds. Therefore, solving the KKT conditions will allow us to solve the problem to optimality.

(e) [5 points] Consider the first two constraints, namely $x_1^2 + 2x_2^2 \leq 10$ and $x_1 + x_3^4 \leq 5$. Can each of them (separately) be represented as a Second-Order Conic constraint ? If yes, show how. If not, motivate your answer.

Constraint $x_1^2 + 2x_2^2 \leq 10$ can be rewritten as follows:

$$x_1^2 + (\sqrt{2}x_2)^2 \leq 10 \longrightarrow \sqrt{x_1^2 + (\sqrt{2}x_2)^2} \leq \sqrt{10} \tag{1}$$

which can be represented in conic form as

$$\begin{pmatrix} x_1 \\ \sqrt{2}x_2 \\ \sqrt{10} \end{pmatrix} \in \boldsymbol{L}^2 \iff A\boldsymbol{x} + \boldsymbol{b} \in \boldsymbol{L}^2.$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{10} \end{pmatrix}.$$

Constraint $x_1 + x_3^4 \leq 5$ can be reformulated as follows:

$$x_1 + (x_3^2)^2 \leq 5 \longrightarrow x_1 + s^2 \leq 5$$

where s and t are additional variables. After adding the constraint $s \geq x_3^2$ (another SOC constraint) and $t = 5 - x_1$ (a simple linear transformation) we can write our constraint as follows:

$$s^2 \leq t$$

that is equivalent to

$$\begin{pmatrix} 2s \\ t-1 \\ t+1 \end{pmatrix} \in \boldsymbol{L}^2 \iff A(s,\ t)^\top + \boldsymbol{b} \in \boldsymbol{L}^2.$$

where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Exercise 4 [23 points = 10 + 7 + 6]

Consider the following optimization problem:

$$\begin{aligned} \min \quad & \max\{3x_1, x_1 + 6x_2\} \\ \text{s.t.} \quad & 2x_1x_2 + x_1 \geq 1 \\ & x_1 - 3x_2 \geq -2 \\ & x_1 \leq 10 \\ & x_1 \geq 0 \\ & x_2 \in \{0, 1\}. \end{aligned}$$

(a) [10 points] Reformulate the problem as an (integer) linear program, modifying both the objective and the nonlinear constraint.

The objective function can be replaced by a new continuous real variable $t \in \mathbb{R}$, provided that we add two new constraints $3x_1 \leq t$ and $x_1 + 6x_2 \leq t$ to the problem.

Focusing on the first constraint, the product x_1x_2 therein can be "linearized" using the fact that $x_2 \in \{0, 1\}$ and $0 \leq x_1 \leq 10$. More specifically, x_1x_2 can be replaced by a new continuous real variable $y \in \mathbb{R}$ that meets the following additional constraints

$$\begin{aligned} & y \leq 10x_2, \\ & y \geq 0, \\ & y \leq x_1, \\ & y \geq x_1 - 10(1 - x_2). \end{aligned}$$

Then, we can rewrite the first inequality as $2y + x_1 \geq 1$. The resulting full ILP then reads as

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 3x_1 \leq t \\ & x_1 + 6x_2 \leq t \\ & 2y + x_1 \geq 1 \\ & x_1 - 3x_2 \geq -2 \\ & 0 \leq x_1 \leq 10 \\ & x_2 \in \{0, 1\}, \\ & y \leq 10x_2, \\ & y \geq 0, \\ & y \leq x_1, \\ & y \geq x_1 - 10(1 - x_2). \end{aligned}$$

Assume now that the coefficients on the left-hand side of the second constraint, i.e., $x_1 - 3x_2 \geq -2$, are now uncertain, and thus we replace it with the following chance constraint

$$\mathbb{P}(z_1x_1 + z_2x_2 \geq -2) \leq 1 - \varepsilon,$$

where $\varepsilon \in (0, 1)$ and $\boldsymbol{z} = (z_1, z_2)$ is the vector of uncertain coefficients.

(b) [7 points] Assuming the vector of uncertain coefficients $\boldsymbol{z} = (z_1, z_2)$ follows a multivariate Gaussian distribution $(\boldsymbol{\mu}, \Sigma)$ with mean $\boldsymbol{\mu} = (1, -3)$ and covariance matrix $\Sigma = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, rewrite the chance constraint above in an equivalent deterministic form.

The chance constraint can be rewritten using the fact that $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$ as

$$\mathbb{P}(z_1x_1 + z_2x_2 < -2) \geq \varepsilon.$$

Using the fact that is a Gaussian distribution, this is equivalent to the constraint

$$\boldsymbol{\mu}^\top \boldsymbol{x} \leq h + \Phi^{-1}(\varepsilon)\sqrt{\boldsymbol{x}^\top \Sigma \boldsymbol{x}}$$

with $h = -2$, $\boldsymbol{\mu} = (1, -3)$, $\Sigma = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, which rewrites as

$$x_1 - 3x_2 \leq -2 + \Phi^{-1}(\varepsilon)\sqrt{2x_1^2 + x_1x_2 + x_2^2},$$

where $\Phi^{-1}(\cdot)$ is the inverse CDF or quantile function of a standard normal distribution.

(c) [6 points] Under the same assumptions as in (b), is the chance constraint introduced above convex for all values of ε? Motivate your answer in either case.

This constraint is convex if and only if $\varepsilon \geq \frac{1}{2}$. Indeed, the function $g(\boldsymbol{x}) := -h + \mu^T\boldsymbol{x} - \Phi^{-1}(\varepsilon)\sqrt{\boldsymbol{x}^T\Sigma\boldsymbol{x}}$ that appears in the constraint $g(\boldsymbol{x}) \leq 0$ is convex because it is the sum of two convex functions, the first one being the affine function $-h + \mu^T\boldsymbol{x}$, the second one being $\Phi^{-1}(\varepsilon)\sqrt{\boldsymbol{x}^T\Sigma\boldsymbol{x}}$, which inherits the convexity from $\sqrt{\boldsymbol{x}^T\Sigma\boldsymbol{x}}$ (proved in (a)), as long as $\Phi^{-1}(\varepsilon) \leq 0$. Since the probability mass is distributed symmetrically with respect to the origin for a standard normal distribution, the quantile $\Phi^{-1}(\varepsilon)$ is less than or equal to 0 only when $\varepsilon \leq 1/2$.

Exercise 5 [25 points = 10 + 10 + 5]

Taif is a company that manufactures cars in two factories and then ships them to three regions in Europe. The two factories, labeled as A and B, can supply at most 450 and 600 cars, respectively. The customer demands in regions 1, 2, and 3 are equal to 450, 200, and 300 cars, respectively. To ship a car from each factory to each region Taif must pay a shipping cost (*unit shipping costs*). This cost is however subject to uncertainty, depending on different unpredictable factors (weather, traffic, truck availability, etc.).

Taif has some data available regarding the *unit shipping costs* of previous shipping. After analyzing such data, Taif was able to derive a minimum and a maximum unit shipping cost from each factory and region pair, as reported in the following table (values in €/car).

	Region 1		Region 2		Region 3	
	min	max	min	max	min	max
Factory A	131	150	218	230	266	270
Factory B	250	280	116	120	263	270

Additionally, Taif observed that the sum of all the 6 *unit shipping costs* (from every possible factory to every possible region) never exceeds a total amount $C_{\max} = 1500$ €/car.

Taif wants to find the lowest-cost shipping plan for meeting the demands of the four regions without exceeding the capacities of the factories in the *worst-case*, taking into consideration the information on the minimum and maximum observed unit shipping cost for each connection and on the sum of all the unit shipping costs C.

(a) [10 points] Formulate a robust mathematical optimization model that Taif can use to solve its shipping cost minimization problem. Introduce and define the decision variables, describe the constraints, distinguishing the deterministic ones and those affected by uncertainty, and explicitly describe the uncertainty set in which the uncertain parameters vary. What type of uncertainty set is it?

Let us first introduce the following sets and parameters

- F is the set of factories: $F = \{A, B\}$
- R is the set of destinations/regions: $R = \{1, 2, 3\}$
- s_i is the availability/supply of factory $i \in F$
- d_j is the demand of destination/region $j \in R$
- z_{ij} is the unit shipping cost from origin $i \in F$ to destination $j \in R$. These parameters are subject to uncertainty and belong to the uncertainty set $\mathcal{Z} \subseteq \mathbb{R}^{2\times 3}$

The uncertainty set $\mathcal{Z}$ is a polyhedral uncertainty set defined by

$$\mathcal{Z} = \left\{ \boldsymbol{z} \in \mathbb{R}^{2\times 3} \ : \ z_{ij} \in [\underline{c}_{ij}, \overline{c}_{ij}] \ \forall i \in F, j \in R, \ \sum_{i\in F}\sum_{j\in R} z_{ij} \leq C_{\max} \right\},$$

where $\underline{c}_{ij}$ and $\overline{c}_{ij}$ represent the values of the minimum and maximum shipping cost from factory i to region j, for each $i \in F$ and $j \in R$, respectively and defined according to the provided table above (e.g., $\underline{c}_{11} = 131$, $\overline{c}_{ij} = 150$).

If you noticed that even in the worst case (that is when the shipping costs all take the highest values), the sum $\sum_{i \in F} \sum_{j \in R} z_{ij} = 1320 < 1500 = C_{\max}$ is redundant and thus be removed from the uncertainty set, which can then be recognized as a simpler box uncertainty set.

Denote by $x_{ij} \in \mathbb{Z}_+$ the decision variables describing the number of cars to ship from factory $i \in F$ to region $j \in R$. The objective function of the problem is to minimize the worst-possibly shipping costs and can thus be written as

$$\min \max_{\boldsymbol{z} \in \mathcal{Z}} \quad \sum_{i \in F} \sum_{j \in R} z_{ij} x_{ij}$$

The following two constraints describe respectively the maximum car production in each factory and the customer demand in each region:

$$\sum_{j \in R} x_{ij} \le s_i \quad \forall i \in F,$$
$$\sum_{i \in F} x_{ij} = d_j \quad \forall j \in R.$$

The uncertainty can be moved from the objective function into a (new) constraint by adding an auxiliary variable $t \in \mathbb{R}$. The resulting robust optimization problem then is:

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & t \ge \sum_{i \in F} \sum_{j \in R} z_{ij} x_{ij} && \forall \boldsymbol{z} \in \mathcal{Z} \\ & \sum_{j \in R} x_{ij} \le s_i && \forall i \in F \\ & \sum_{i \in F} x_{ij} = d_j && \forall j \in R \\ & x_{ij} \in \mathbb{Z}_+ && \forall i \in F, j \in R. \end{aligned}$$

(b) [10 points] Identify the adversarial problem of your robust model, define its dual, and derive the tractable robust counterpart of the problem.
Hint: First use one of the standard tricks to move the uncertainty from the objective to a constraint.

The adversarial problem is:

$$\begin{aligned} \max \quad & \sum_{i \in F} \sum_{j \in R} z_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i \in F} \sum_{j \in R} z_{ij} \le C_{\max} && [\tau] \\ & z_{ij} \le \bar{c}_{ij} \qquad \forall i \in F, \forall j \in R && [u_{ij}] \\ & z_{ij} \ge \underline{c}_{ij} \qquad \forall i \in F, \forall j \in R && [\ell_{ij}] \end{aligned}$$

Using standard dual theory or the formula for polyhedral uncertainty, we can show that the dual of the adversarial problem is:

$$\begin{aligned} \min \quad & \tau C_{\max} + \sum_{i \in F} \sum_{j \in R} (u_{ij} \bar{c}_{ij} + \ell_{ij} \underline{c}_{ij}) \\ \text{s.t.} \quad & \tau + u_{ij} + \ell_{ij} = x_{ij} && \forall i \in R, j \in F \quad [z_{ij}] \\ & \tau \ge 0 && \forall i \in F, \forall j \in R \\ & u_{ij} \ge 0 && \forall i \in F, \forall j \in R \\ & \ell_{ij} \le 0 && \forall i \in F, \forall j \in R. \end{aligned}$$

The tractable reformulation of the robust problem then is

$$
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & t \geq \tau C_{\max} + \sum_{i \in F} \sum_{j \in R} \left(u_{ij} \bar{c}_{ij} + \ell_{ij} \underline{c}_{ij} \right) \\
& \tau + u_{ij} + \ell_{ij} = x_{ij} && \forall i \in R, j \in F \\
& \sum_{j \in R} x_{ij} \leq s_i && \forall i \in F \\
& \sum_{i \in F} x_{ij} = d_j && \forall j \in R \\
& x_{ij} \in \mathbb{Z}_+ && \forall i \in F, j \in R \\
& \tau \geq 0 && \forall i \in F, \forall j \in R \\
& u_{ij} \geq 0 && \forall i \in F, \forall j \in R \\
& \ell_{ij} \leq 0 && \forall i \in F, \forall j \in R.
\end{aligned}
$$

If you correctly derived a simpler box uncertainty set in part (a), the solution you should have obtained is the one without the terms τ and $\tau C_{\max}$.

(c) [5 points] Assume that the maximum value of the total sum of the 6 *unit shipping costs* is decreased to $C_{\max} = 1400$ €/car. What is the relationship between the optimal solution cost of the new robust problem (obtained setting $C_{\max} = 1400$) and the solution of the previously formulated robust problem ($C_{\max} = 1500$)?

We can conclude that the following inequality holds for the optimal value of the primal problem in the two cases:

$$v(P)_{C_{\max}=1400} \leq v(P)_{C_{\max}=1500}.$$

This inequality is a consequence of the fact that the uncertainty set is possibly reduced when $C_{\max} = 1400$ compared to the case $C_{\max} = 1500$.

If you correctly spotted the redundancy and derived a simpler box uncertainty set in part (a), you could have further concluded that the optimal values are actually identical since the two uncertainty sets coincide.