vrije Universiteit amsterdam

Faculty of Exact Sciences

Exam:	Mathematical Optimization
Code:	XM_0051
Examiner:	Joaquim Gromicho
Co-reader:	Alessandro Zocca
Date:	5 February 2020
Time:	18:30
Duration:	2 hours and 45 minutes
Calculator allowed:	Yes
Graphical calculator allowed:	No
Number of questions:	14 in 5 question groups.
Type of questions:	Open
Answer in:	**English** (note that Dutch will not be corrected!!!)

Remarks:

- A solution without explanation is considered wrong.
- The credits that each individual question is worth are displayed next to it.
- Remember to answer in English!

Credit score:	10.0 = 1.0 + (0.5+1.0+0.5+1.0) + 1.0 + (0.5+0.5+0.5+0.5+0.5+1.0+0.5) + 1.0
Grades:	before Wednesday February 19, 2020.
Inspection:	Upon request.
Number of pages:	3 including cover page

Good luck!

1. (*ILO modeling*)	1 credit

Suppose that we want to formulate an optimization problem on the decision variables $\boldsymbol{x} \in R^n$ that involves two constraints of the form $\boldsymbol{d}^T\boldsymbol{x} \leq f$ and $\boldsymbol{g}^T\boldsymbol{x} \leq h$ for two vectors $\boldsymbol{d}, \boldsymbol{g} \in R^n$ and two real numbers f, h.

The underlying model at hand has the following property: if the first constraint is satisfied, then the second is not needed. In other words, the constraint $\boldsymbol{g}^T\boldsymbol{x} \leq h$ must only hold if $\boldsymbol{d}^T\boldsymbol{x} \leq f$ is violated. Explain how you would formally model this.

2. (*MO & optimality*)	3 credits

Consider the following mathematical optimization problem expressed in two variables:

$$\begin{aligned} \max \quad & xy \\ \text{st:} \quad & x + y^2 \leq 1 \\ & x, y \geq 0 \end{aligned}$$

2.1 *(0.5 credits)* Is it a convex optimization problem?

2.2 *(1 credit)* Express the Karush-Kuhn-Tucker conditions.

2.3 *(0.5 credits)* Show that that $x = y = \lambda_1 = \lambda_2 = \lambda_3 = 0$ satisfies the KKT conditions. Is this an optimal solution for the problem?

2.4 *(1 credit)* Use the KKT conditions to solve the problem.

3. (*CQr*)	1 credit

Convert the constraint $x + y^2 \leq 1$ into a conic quadratic inequality $||\boldsymbol{Du} + \boldsymbol{d}||_2 \leq \boldsymbol{pu} + q$ with $\boldsymbol{u} = (x, y)$ for the appropriate matrix $\boldsymbol{D}$, vectors $\boldsymbol{d}$ and $\boldsymbol{p}$ and scalar q. You may use as many second order conic inequalities as you need.

Hint: remember that $a = \frac{(a+1)^2}{4} - \frac{(a-1)^2}{4}$.

4. Lagrange Duality for MILO	4 credits

Consider the capacitated location model described below. This model is defined as follows:

$$
\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{j\in J} h_j y_j \\
\text{st:} \quad & \sum_{j\in J} x_{ij} = 1 && \forall i \in I \quad (4.a) \\
& \sum_{i\in I} d_i x_{ij} \le u_j y_j && \forall j \in J \quad (4.b) \\
& 0 \le x_{ij} \le 1 && \forall i \in I \quad \forall j \in J \\
& y_j \in \{0,1\} && \forall j \in J
\end{aligned}
$$

Given a set of customers I and a set of locations J, one interpretation for this model is the following:

- x_{ij} is the proportion of demand d_i of customer i served at facility j with cost of service c_{ij}
- facility j costs h_j to be installed and open for service offering capacity u_j
- each customer must satisfy the full demand $(4.a)$
- open facilities offer capacity $(4.b)$

The objective of the problem is to decide which facilities to open and how to satisfy the customer's demand at minimum cost. Assume that all values c_{ij}, h_j, d_i and u_j are positive and integer and answer the following questions:

4.1 (*0.5 credits*) Is the constrain matrix of the whole model Total Unimodular in general?

4.2 (*0.5 credits*) Show that the part of the constraint matrix that defines only constraints $(4.a)$ is Totally Unimodular.

4.3 (*0.5 credits*) Describe the problem that you obtain when you **ignore** $(4.b)$ and explain how you can solve it analytically.

4.4 (*0.5 credits*) Describe the problem - and its solution - that you obtain when you **ignore** $(4.a)$.

4.5 (*0.5 credits*) Describe the problem that you obtain when you **relax** $(4.a)$ and explain how you would solve it. Note that relaxing (in the Lagrangian way) modifies the costs.

4.6 (*1.0 credit*) Suppose that you could freely choose which Lagrange Dual to solve: either relaxing constraints $(4.a)$ or relaxing constraints $(4.b)$. Which of these two would you prefer and why?

4.7 (*0.5 credits*) Remember the uncapacitated facility location model, for which you learned a weak and a strong model. The model given here for the capacitated case is like the weak model of the uncapacitated case. Suggest a way to strengthen it.

5. (*RO*)	1 credit

Starting from the optimization problem in part 4, suppose that the demand is now uncertain but known to be inside a ball centered at the nominal vector $\boldsymbol{d} = [d_1 \quad \cdots \quad d_i \quad \cdots \quad d_{\#I}]$. Furthermore, suppose that the radius of this uncertainty ball depends on the facility, i.e., is equal to $\delta_j > 0$ for each $j \in J$. Explain how to model the robust version of the problem by giving the robust counterpart of the given uncertainty set.