vrije Universiteit amsterdam

Faculty of Exact Sciences

Exam:	Mathematical Optimization
Code:	XM_0051
Examiner:	Joaquim Gromicho
Co-reader:	Alessandro Zocca
Date:	5 February 2020
Time:	18:30
Duration:	2 hours and 45 minutes
Calculator allowed:	Yes
Graphical calculator allowed:	No
Number of questions:	14 in 5 question groups.
Type of questions:	Open
Answer in:	**English** (note that Dutch will not be corrected!!!)

Remarks:

- A solution without explanation is considered wrong.
- The credits that each individual question is worth are displayed next to it.
- Despite the fact that multiple questions concern the same optimization problem, the questions are posed in such a way that failure to answer one question does not affect your ability to solve another. You may use the statements made presented in one (unanswered) questions to answer any of the remaining questions.

Credit score:	10.0 = 1.0 + (0.5+1.0+0.5+1.0) + 1.0 + (0.5+0.5+0.5+0.5+0.5+1.0+0.5) + 1.0
Grades:	before Wednesday February 19, 2020.
Inspection:	Upon request.
Number of pages:	6 including cover page

Good luck!

1. (*ILO modeling*)	1 credit

Suppose that we want to formulate an optimization problem on the decision variables $\boldsymbol{x} \in R^n$ that involves two constraints of the form $\boldsymbol{d}^T\boldsymbol{x} \leq f$ and $\boldsymbol{g}^T\boldsymbol{x} \leq h$ for two vectors $\boldsymbol{d}, \boldsymbol{g} \in R^n$ and two real numbers f, h.

The underlying model at hand has the following property: if the first constraint is satisfied, then the second is not needed. In other words, the constraint $\boldsymbol{g}^T\boldsymbol{x} \leq h$ must only hold if $\boldsymbol{d}^T\boldsymbol{x} \leq f$ is violated. Explain how you would formally model this.

0.5 Note that $d^T x > f \;\Rightarrow g^T x \leq h$ is equivalent to $d^T x \leq f \;\vee\; g^T x \leq h$. Therefore, we need to model that either the first or the second constraint are satisfied.
0.5 If we may find a sufficiently large M such that adding it to the right-hand side makes the corresponding constraint redundant, then it is enough to add a binary variable $y \in \{0,1\}$ to 'decide' which constraint must hold, namely

$$\begin{aligned} d^T x &\leq f + My \\ g^T x &\leq h + M(1-y) \end{aligned}$$

2. (*MO & optimality*)	3 credits

Consider the following mathematical optimization problem expressed in two variables:

$$\begin{aligned} \max \quad & xy \\ \text{st:} \quad & x + y^2 \leq 1 \\ & x, y \geq 0 \end{aligned}$$

2.1 *(0.5 credits)* Is it a convex optimization problem?

0.25 The feasible region is clearly defined by 'convex inequalities', since affine functions are convex and the sum of convex functions is convex. The objective function $f(x,y)$ has as gradient $\nabla f = [y \quad x]$ and as Hessian $\nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
0.25 This matrix is clearly indefinite (eigenvalues $+1, -1$, eigenvectors $(-1,1)$, $(1,1)$, leading principal minors 0 and -1, etc), making the objective function neither convex nor concave. Thus, it is not a convex optimization problem.

2.2 *(1 credit)* Express the Karush-Kuhn-Tucker conditions.

The KKT conditions are always necessary for optimality, and we can express them regardless of the convexity of the problem.
0.5 In order to have our problem in the usual minimization form we consider this equivalent formulation:

$$\begin{aligned} \min \quad & -xy \\ \text{st:} \quad & x + y^2 - 1 \leq 0 \\ & -x, -y \leq 0 \end{aligned}$$

Setting

$$\begin{aligned} f(x,y) &= -xy \\ g_1(x,y) &= x + y^2 - 1 \\ g_2(x,y) &= -x \\ g_3(x,y) &= -y \end{aligned}$$

we obtain

$$\nabla f(x,y) = [-y \quad -x]$$
$$\nabla g(x,y) = \begin{bmatrix} 1 & 2y \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

0.5 Introducing the non-negative dual variables for $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, the KKT conditions then are:

$$-y + \lambda_1 - \lambda_2 = 0$$
$$-x + 2y\lambda_1 - \lambda_3 = 0$$
$$\lambda_1(x + y^2 - 1) = 0$$
$$\lambda_2(-x) = 0$$
$$\lambda_3(-y) = 0$$
$$x + y^2 - 1 \leq 0$$
$$-x \leq 0$$
$$-y \leq 0$$
$$\boldsymbol{\lambda} \geq \mathbf{0}$$

Alternative solution Leaving the problem as a maximization, all the constraints should be inequalities of the form ≥ 0

$$f(x,y) = xy$$
$$g_1(x,y) = -x - y^2 + 1$$
$$g_2(x,y) = x$$
$$g_3(x,y) = y$$

We obtain therefore:

$$\nabla f(x,y) = [y \quad x]$$
$$\nabla g(x,y) = \begin{bmatrix} -1 & -2y \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Introducing the non-positive dual variables for $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, the KKT conditions then are:

$$y - \lambda_1 + \lambda_2 = 0$$
$$x - 2y\lambda_1 + \lambda_3 = 0$$
$$\lambda_1(-x - y^2 +) = 0$$
$$\lambda_2(x) = 0$$
$$\lambda_3(y) = 0$$
$$-x - y^2 + 1 \geq 0$$
$$x \geq 0$$
$$y \geq 0$$
$$\boldsymbol{\lambda} \leq \mathbf{0}$$

which coincides with the above ones.

2.3 *(0.5 credits)* Show that that $x = y = \lambda_1 = \lambda_2 = \lambda_3 = 0$ satisfies the KKT conditions. Is this an optimal solution for the problem?

0.25 We can easily see that $x = y = \lambda_1 = \lambda_2 = \lambda_3 = 0$ is a point satisfying the KKT conditions.
0.25 This point is clearly not a maximum: for this point the objective function yields the value 0, while for instance $(1,0)$ and $(0,1)$ are both feasible and yield a strictly larger value, namely 1.
[Half the grade, that is 0.25, can be obtained for this question without the KKT conditions only by showing that $(0,0)$ cannot be optimal.]

2.4 *(1 credit)* Use the KKT conditions to solve the problem.

0.4 (Case separation) Note that $\lambda_1 = 0$ implies from the first two conditions that $y + \lambda_2 = x + \lambda_3 = 0$ which is only possible if all are zero (due to the nonnegativity). That is the case of the previous question. Therefore, we must now consider the case $x + y^2 = 1$ meaning that $x = 1 - y^2$ and x and y no longer can be both 0. Let us consider the two resulting cases:

If $x > 0$, then $\lambda_2 = 0$ (complementary slackness, 4th condition) and $\lambda_1 = y$ (1st condition) leading to $2y^2 = x + \lambda_3$ (second condition) which is clearly $\geq x > 0$. Therefore $y > 0$ and $y_3 = 0$ (complementary slackness, 5th condition) leading to $y = \sqrt{\frac{x}{2}}$. Since $\lambda_1 = y > 0$ the 3rd condition gives $x + \frac{x}{2} = 1$, yielding $x = \frac{2}{3}$ and $y = \frac{\sqrt{3}}{3}$.

If $x = 0$, then since we are also assuming $\lambda_1 > 0$ we must have $y^2 = 1$ (complementary slackness, 3rd condition) meaning that $y > 0$ and $\lambda_3 = 0$ (complementary slackness, 5th condition) which reduces the 2nd condition to $2y\lambda_1 = 0$ leading to $\lambda_1 = 0$ and, hence, back to the case of the previous question.

0.4 (Correct solutions) To conclude, the only two solutions of the KKT system are: $(0,0)$ and $\left(\frac{2}{3}, \frac{\sqrt{3}}{3}\right)$.
0.2 (Identification of the maximum) In view of the previous question, it immediately follows that the second must be optimal, with value $\frac{2\sqrt{3}}{9}$.

3. (*CQr*)	1 credit

Convert the constraint $x + y^2 \leq 1$ into a conic quadratic inequality $\|\boldsymbol{D}\boldsymbol{u} + \boldsymbol{d}\|_2 \leq \boldsymbol{p}\boldsymbol{u} + q$ with $\boldsymbol{u} = (x, y)$ for the appropriate matrix $\boldsymbol{D}$, vectors $\boldsymbol{d}$ and $\boldsymbol{p}$ and scalar q. You may use as many second order conic inequalities as you need.

Hint: remember that $a = \frac{(a+1)^2}{4} - \frac{(a-1)^2}{4}$.

0.5 (correct derivation) + 0.5 (final matrix form) A possible CQr reformulation can be obtained as follows:

$$x + y^2 \leq 1$$

$$y^2 \leq 1 - x$$

$$y^2 \leq \frac{(1 - x + 1)^2}{4} - \frac{(1 - x - 1)^2}{4}$$

$$y^2 + \frac{x^2}{4} \leq \frac{(2 - x)^2}{4}$$

$$(2y)^2 + x^2 \leq (2 - x)^2$$

$$\sqrt{x^2 + (2y)^2} \leq 2 - x$$

$$\left\| \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \right\|_2 \leq \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 2$$

Note that there are other possible valid solutions for matrix $\boldsymbol{D}$, vectors $\boldsymbol{d}$ and $\boldsymbol{p}$ and scalar q.

4. Lagrange Duality for MILO	4 credits

Consider the capacitated location model described below. This model is defined as follows:

$$\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} + \sum_{j\in J} h_j y_j \\
\text{st:} \quad & \sum_{j\in J} x_{ij} = 1 && \forall i \in I \quad (4.a) \\
& \sum_{i\in I} d_i x_{ij} \le u_j y_j && \forall j \in J \quad (4.b) \\
& 0 \le x_{ij} \le 1 && \forall i \in I \quad \forall j \in J \\
& y_j \in \{0,1\} && \forall j \in J
\end{aligned}$$

Given a set of customers I and a set of locations J, one interpretation for this model is the following:

- x_{ij} is the proportion of demand d_i of customer i served at facility j with cost of service c_{ij}
- facility j costs h_j to be installed and open for service offering capacity u_j
- each customer must satisfy the full demand $(4.a)$
- open facilities offer capacity $(4.b)$

The objective of the problem is to decide which facilities to open and how to satisfy the customer's demand at minimum cost. Assume that all values c_{ij}, h_j, d_i and u_j are positive and integer and answer the following questions:

4.1 (*0.5 credits*) Is the constrain matrix of the whole model Total Unimodular in general?

0.5 Clearly not, since d_i and u_j can possibly (and most likely) be bigger than 1.

4.2 (*0.5 credits*) Show that the part of the constraint matrix that defines only constraints $(4.a)$ is Totally Unimodular.

0.5 Any submatrix consists only of $0'$s and $1'$s. Each column has at most one 1. If a row exists with only 0 then the determinant is 0. If two columns are equal, then the determinant is also 0. In the remainder case the columns can be rearranged into an identity matrix, meaning that the determinant was either 1 or -1.

4.3 (*0.5 credits*) Describe the problem that you obtain when you **ignore** $(4.b)$ and explain how you can solve it analytically.

0.5 The problem can be separated in one problem for every i. For each of these subproblems indexed by i we select the j with minimum c_{ij}.

4.4 (*0.5 credits*) Describe the problem - and its solution - that you obtain when you **ignore** $(4.a)$.

0.5 Ignoring $(4.a)$ makes the solution $x_{ij} = 0 = y_j$ feasible, which is optimal.

4.5 (*0.5 credits*) Describe the problem that you obtain when you **relax** $(4.a)$ and explain how you would solve it. Note that relaxing (in the Lagrangian way) modifies the costs.

0.5 The solution $x_{ij} = 0 = y_j$ is feasible but may no longer be optimal since the coefficients of x_{ij} in the objective function of the relaxation may become negative. The problem separates in $\#J$ independent problems each with the structure, where $\widetilde{c_{ij}}$ is modified by the relaxation:

$$\begin{aligned} \min \quad & \sum_{i \in I} \widetilde{c_{ij}} x_{ij} + h_j y_j \\ \text{st:} \quad & \sum_{i \in I} d_i x_{ij} \le u_j y_j \\ & 0 \le x_{ij} \le 1 \qquad \forall i \in I \\ & y_j \in \{0,1\} \end{aligned}$$

If all $\widetilde{c_{ij}}$ are positive, the solution is $x_{ij} = 0 = y_j$ otherwise we compute the solution with $y_j = 1$ and take it if its value is negative. The solution for $y_j = 1$ can be found by considering only the i for which $\widetilde{c_{ij}}$ is negative and noticing that we have in fact a fractional knapsack. This means that the problem can be solved easily *[it is fine if the student does not comment on this]*.

4.6 (*1.0 credits*) Suppose that you could freely choose which Lagrange Dual to solve: either relaxing constraints $(4.a)$ or relaxing constraints $(4.b)$. Which of these two would you prefer and why?

0.2 Relaxing constraints $(4.b)$ leads to a problem that is naturally integer (as shown in **4.2**). 0.3 This means that the corresponding dual has the **same value** as the linear relaxation of the original problem, which is the weakest value a Lagrange dual may yield. 0.5 Relaxing $(4.a)$ is therefore preferable, since the corresponding dual may have a value closer to the optimum of the problem.

4.7 (*0.5 credits*) Remember the uncapacitated facility location model, for which you learned a weak and a strong model. The model given here for the capacitated case is like the weak model of the uncapacitated case. Suggest a way to strengthen it.

0.5 Adding $x_{ij} \le y_j$ for every i and j.

5. (*RO*)	1 credit

Suppose that the demand is now uncertain but known to be inside a ball centered at the nominal vector $\boldsymbol{d} = [d_1 \quad \cdots \quad d_i \quad \cdots \quad d_{\#I}]$. Furthermore, suppose that the radius of this uncertainty ball depends on the facility, i.e., is equal to $\delta_j > 0$ for each $j \in J$. Explain how to model the robust version of the problem by giving the robust counterpart of the given uncertainty set.

0.5 (correct dual problem) + 0.5 (correct final robust constraint) The constraint $(4.b)$ simply becomes $\sum_{i \in I} d_i x_{ij} + \delta_j \|x_{.j}\|_2 \le u_j y_j$ with $x_{.j} = [x_{1j} \quad \cdots \quad x_{ij} \quad \cdots \quad x_{\#Ij}]$.