ReExam Mathematical Economics 3, May 2021

Question 1: Games and Networks (24 points)

Consider the sequencing problem (N, p, q, ρ) with three agents $N = \{1, 2, 3\}$ with linear cost functions $q_1(t) = 5t$, $q_2(t) = 2t$ and $q_3(t) = 10t$, processing times $p_1 = p_2 = p_3 = 1$, and the initial positions in the queue given by $\rho(i) = i,\ i \in \{1, 2, 3\}$.

a. (8 points) Give the corresponding sequencing game (N, v). (If you prefer, you can give the Harsanyi dividends $\Delta_v(S), S \subseteq N$, instead of the characteristic function v.)

b. (8 points) Compute $m^{\rho^u}(v, L)$, $m^{\rho^\ell}(v, L)$ and the Shapley value outcome of the sequencing game (N, v) of question a. Hint: Notice that $m^{\rho^u}(v, L)$ (respectively $m^{\rho^\ell}(v, L)$) is equal to the hierarchical outcome corresponding to root player $n = 3$ (respectively $n = 1$.)

Show that all three payoff vectors belong to the Core of the sequencing game.

Now, consider an arbitrary sequencing game (N, v).

c. (8 points) Argue that this game is superadditive.

Question 2: Games and Networks (8 points)

Prove that the Shapley value φ is an efficient solution for TU-games. In other words, prove that $\sum_{i \in N} \varphi_i(N, v) = v(N)$ for every TU-game (N, v).

Question 3: Search and Matching (16 points)

a. (8 points) Consider the value of employment in the Pissarides model in discrete time.

$$V^E = w + \frac{1}{1+r}\left[(1-\delta)V^E + \delta V^U\right] \tag{1}$$

where y is output, w is the wage, r is the interest rate/discount rate and δ is the job destruction rate. Explain this equation in words, take the continuous time limit and derive the continuous time value function.

b. (8 points) How would you adjust the Diamond-Mortensen-Pissarides model to study the Covid 19 recession?

Question 4: Collusion and Competition Policy (16 points)

Assume that the market is dominated by two symmetric firms. The two firms offer symmetrically differentiated products and compete in prices. The firms' demand functions are $q_1 = 6 - 2p_1 + p_2$ and $q_2 = 6 - 2p_2 + p_1$, respectively. Both firms have constant marginal costs that are normalized to zero, i.e. $c_1 = c_2 = 0$. q_1 and q_2 denote quantities of firm 1 and 2, respectively. p_1 and p_2 are the prices of the differentiated products.

a. (4 points) Find best-response functions of both firms. Determine Nash equilibrium prices and profits in this price-setting game.

Next, consider settings where each of the two firms can choose between either colluding and sharing monopoly profits $(\pi^M/2,\ \pi^M/2)$, or competing and receiving Nash profits.

b. (6 points) Provide the matrix form representation of the simultaneous moves game described above. Identify (graphically) in the (π_1, π_2)-diagram the set of feasible payoffs and the set of sustainable payoffs in this matrix form game. Recall each firm has two possible actions: collude or deviate.

c. (6 points) Next, assume that the price-competition game is played repeatedly for infinitely many periods with discount factor δ. Firms use grim-trigger strategies: each firm sets the collusive (monopoly) price $p^M = 3$ in every period if there were no deviation in any of the previous periods. In case of a deviation, firms revert to the punishment phase with static Nash payoffs in every period after the deviation is observed. Write down the incentive compatibility constraint for collusion to be sustained in this industry. Determine the smallest value of the discount factor for which collusion on monopoly price constitutes a Sub-game perfect Nash equilibrium.

Question 5: Auctions (16 points)

Consider the *first-price-squared* sealed bid auction. In this auction, bidders are invited to submit a sealed bid, with the understanding that the item will be allocated to the highest bidder, for a price equal to the square of her bid (i.e., her bid to the power 2).

Suppose there are $n = 2$ bidders with values for the item v drawn from the interval $[0, 1]$ according to the uniform distribution.

Take the position of bidder 1. Suppose bidder 1 learns that bidder 2 will bid according to the function $b_2(v) = \sqrt{\frac{v}{\alpha}}$, where α is a positive scalar.

a. (6 points) Derive the best-response of bidder 1.

b. (8 points) Derive the symmetric Bayes-Nash equilibrium of the *first-price-squared* sealed bid auction.

c. (1 point) Without making calculations. Does this auction raise more funds than the standard auctions we have seen in class (first- and second-price sealed bid auctions)? Why?

d. (1 point) Is this auction efficient from an economic point of view. If not, why? If yes, why?

Question 6: Market Design in Practice (16 points)

The tennis club is having a lottery for its 14 members. They have five items for the lottery, a bike, a tennis racket, a set of tennis balls, a shirt and a free membership for next season. Jim is a member of the tennis club and his valuations for the different items are stated in the second column of the table below.

	value	interest
bike	120	4
tennis racket	50	3
set of balls	15	1
shirt	25	0
free membership	130	5

The lottery works in rounds. For the first round all members of the tennis club get one lottery ticket. They can decide to use their ticket for only one item. Then the items are randomly assigned to a member that placed a lottery ticket on the item. If no member placed a lottery ticket on a specific item, this item moves to the next round. For this next round all members that did not win in the first round get a new lottery ticket. And this next round works the same as the first round.

Jim is risk neutral and mainly interested in obtaining the highest expected value from the lotteries. He figures out on which item the other 13 members are placing their lottery ticket. This is shown in the third column of the table above.

a. (8 points) On which item should Jim place his lottery ticket in the first round? Explain your answer.

b. (4 points) Is the lottery mechanism strategy proof? Explain your answer.

c. (4 points) Explain how the lottery mechanism would take place if the organizers would follow random serial dictatorship.

Short answers of the Exam Mathematical Economics 3, May 2021

1.a. The sequencing game is
$v(\{i\}) = v_L(i) = 0$ for all $i \in N$.
$v(\{1,2\}) = \max(0, 5 + (2 \cdot 2) - 2 - (2 \cdot 5)) = 0$
$v(\{1,3\}) = 0 + 0 = 0$
$v(\{2,3\}) = \max(0, (2 \cdot 2) + (3 \cdot 10) - (2 \cdot 10) - (3 \cdot 2)) = 8$
$v(\{1,2,3\}) = \max(0, 5 + (2 \cdot 2) + (3 \cdot 10) - 10 - (2 \cdot 5) - (3 \cdot 2)) = 13.$

Alternatively,
$\Delta_v(\{i\}) = \Delta_{v_L}(i) = 0$ for all $i \in N$.
$\Delta_v(\{1,2\}) = \max(0, 2 - 5) = 0$
$\Delta_v(\{1,3\}) = 0$
$\Delta_v(\{2,3\}) = \max(0, 10 - 2) = 8$
$v(\{1,2,3\}) = \max(0, 10 - 5) = 5.$

1.b. $m^{\rho^u}(v, L) = h^n(v, L) = (0, 0, 13)$, $m^{\rho^\ell}(v, L) = h^1(v, L) = (5, 8, 0)$, $\varphi(v) = (\frac{5}{3}, \frac{8}{2} + \frac{5}{3}, \frac{8}{2} + \frac{5}{3}) = \frac{1}{3}(5, 17, 17)$.
The Core consists of all payoff vectors that allocate 13 in such a way that (i) every job gets a nonnegative payoff and (ii) jobs 2 and 3 get at least 8 together. This is satisfied by all three payoff vectors.

1.c. Consider two disjoint consecutive coalitions $S = [i; j]$ and $T = [h : m]$ with $j < h$. If $h > j + 1$ then $S \cup T$ is not connected, and since $v = v_L$, the worth of $S \cup T$ is the sum of the worth of its components S and T, i.e. $v(S) + v(T) = v(S) + v(T)$.
If $h = j + 1$ then $S \cup T$ is connected. But then the agents in $S \cup T$ can anyway reorder themselves in such a way that first the agents in S are served (in their efficient order), and then the agents in T (in their efficient order). This gives cost saving $v(S) + v(T)$. But cost saving can be higher since agents in S can switch position with agents in T. So, at least the cost saving will be equal to $v(S) + v(T)$, i.e. $v(S \cup T) \geq v(S) + v(T)$. Thus, the sequencing game is superadditive.

2. See slides, book and exercises

3.a. Worker receives wage and in the next period (which is discounted), he remains employed with probability $(1 - \delta)$ and flows into unemployment with probability δ.

$$\begin{aligned} V^E &= \Delta w + \frac{1}{1 + r\Delta}\left[(1 - \delta\Delta)V^E + \delta\Delta V^U\right] && (2) \\ (1 + r\Delta - 1)\, V^E &= (1 + r\Delta)\,\Delta w - \delta\Delta\left[V^E - V^U\right] && (3) \\ rV^E &= (1 + r\Delta)\, w - \delta\left[V^E - V^U\right] && (4) \\ \lim_{\Delta \to 0} rV^E &= w - \delta\left[V^E - V^U\right] && (5) \end{aligned}$$

3.b. Different answers possible. Combining it with a SIR model, just shocking y. Study UI benefit extensions, allowing for different sectors with lots of contacts and few contacts and or possibilities to work at home.

4.a. Standard best-response analysis implies following best response functions: $p_1(p_2) = (6 + p_2)/4$ and $p_2(p_1) = (6 + p_1)/4$.
This implies that $p_1^* = p_2^* = 2$ and profits are given by $\pi_1^* = \pi_2^* = 8$.

4.b. Profit maximizing prices when two firms act as a single monopolist are given by $p_1^M = p_2^M = 3$. Monopoly profits are $\pi_1 = \pi_2 = 9$.
Matrix form representation of the simultaneous move game described above is as follows

		Firm 2	
		Cooperate	Deviate
Firm 1	Cooperate	$9, 9$	$6\frac{3}{4}, 10\frac{1}{8}$
	Deviate	$10\frac{1}{8}, 6\frac{3}{4}$	$8, 8$

The (π_1, π_2)-diagram identifying the set of feasible payoffs and the set of sustainable payoffs is similar to the diagram presented in the slide 34 of Lecture I (week 5).
NE of this simultaneous moves game is (Deviate, Deviate) with payoffs ($\pi_1^* = 8$, $\pi_2^* = 8$).
The set of feasible payoffs is given by the diamond area, i.e. the convex combinations of the per-period pure strategy payoffs. The set of sustainable payoffs is given by the green area.

4.c. If simultaneous moves game in (a) is repeated infinitely many times, Folk Theorem result implies that any outcome in the set of sustainable payoffs identified in (a) can arise as a SPNE of this infinitely repeated one-stage game.
The constraint for collusion on monopoly price (i.e. $p_1 = p_2 = 3$ and $\pi_1 = \pi_2 = 9$) to be sustainable as a SPNE in the industry with 2 firms is as follows $V^{collusion} \geq V^{deviation}$, or

$$\frac{9}{1-\delta} \geq 10\frac{1}{8} + \frac{8\delta}{1-\delta}.$$

From this inequality, we can obtain the critical discount factor for collusion to be sustainable

$$\delta \geq \frac{81-72}{81-4} = \frac{9}{17} \approx 0.5294$$

5.a.
The best-response function follows from maximising

$$u\left(b_1; b_2(v) = \sqrt{v_2/\alpha}\right) = \Pr\left[b_1 \geq \sqrt{v_2/\alpha}\right](v_1 - b_1^2) = \alpha b_1^2(v_1 - b_1^2)$$

Taking the FOC and solving for b_1 gives

$$b_1 = \sqrt{v_2/2}.$$

5.b.

$$u(b_1; b(v_2)) = \Pr\left[b_1 \geq b(v_2)\right](v_1 - b_1^2) = b^{-1}(b_1)(v_1 - b_1^2)$$

Taking the FOC and applying symmetry (see class slides) gives

$$v = b(v)^2 + 2b(v)b'(v)v$$

Because $\left(b(v)^2 v\right)' = 2b(v)b'(v)v + b(v)^2$, integration on the LHS and RHS of the previous equation gives the equilibrium bidding function:

$$b^*(v) = \sqrt{v/2}.$$

5.c. The auction is *standard* because the item is allocated to the highest bidder and the payment of the bidder with the lowest valuation is zero. Therefore, the revenue equivalence theorem applies, which implies that the revenues is the same as in any other standard auction (including first-price and second-price auctions).

5.d. It is efficient because the item is always allocated to a bidder and it is allocated to the bidder who values it most.

6.a. To be added