Exam Mathematical Economics 3, March 2021

Question 1: Games and Networks (24 points)

Consider a market with one seller, say player 1, who owns a house that is for sale. There are two potential buyers for the house, players 2 and 3. Player 2 has valuation 4 for the house, while player 3 has valuation 8 for the house. The house can only be sold through a real estate agent, player 4. The seller has reservation value zero for the good. Also the valuation of the real estate agent for the good is zero.

This situation can be modeled by a graph game (v, L) on player set $N = \{1, 2, 3, 4\}$, with graph L given by $L = \{\{1, 4\}, \{2, 4\}, \{3, 4\}\}$ and characteristic function v given by

$$v(S) = \begin{cases} 4 & \text{if } S \in \{\{1,2\},\{1,2,4\}\} \\ 8 & \text{if } S \in \{\{1,3\},\{1,2,3\},\{1,3,4\},\{1,2,3,4\}\} \\ 0 & \text{otherwise} \end{cases}$$

a. (10 points) Give the Myerson restricted game v_L, and compute the Myerson value of this game. Hint: To write down the restricted game v_L it is sufficient to mention which coalitions have a nonzero worth, and what is their worth.

b. (10 points) Compute the four hierarchical outcomes of this game (so, compute $h^i(v, L), i \in \{1, 2, 3, 4\}$), and verify if they belong to the Core of the Myerson restricted game v_L.

Suppose that the valuation of buyer 2 decreases to an unknown number $0 < v_2 < 4$.

c. (4 points) Argue that for every value $0 < v_2 < 4$, the Myerson value does not belong to the Core of the restricted game v_L.

Question 2: Games and Networks (8 points)

Let $\Omega \subseteq 2^N$ be the collection of connected coalitions in an undirected graph (N, L). Show that Ω satisfies union stability. (In other words, show that $S, T \in \Omega$ with $S \cap T = \emptyset$ implies that $S \cup T \in \Omega$.)

Question 3: Search and Matching (16 points)

(a) **Bargaining (Binmore, Rubinstein Wolinsky)** Consider the bargaining game of Binmore, Rubinstein and Wolinsky (BRW).

Let z be the outside option of the firm and b the outside option of the worker and y match output. If an offer is rejected, a breakdown occurs with probability p. Assume that there is enough surplus for trade to take place so that the unique SPE is immediate agreement. Let x_w^* be the highest wage that a firm would accept and x_f^*, the lowest wage that a worker would accept.

a. (4 points) Explain briefly in words (max 6 lines), why the equilibrium wage offers under non-transferable utility in BRW can be written as,

$$\begin{aligned} y - x_w^* &= pz + (1-p)\left(y - x_f^*\right) \\ x_f^* &= pb + (1-p)x_w^* \end{aligned}$$

(b) **From discrete to continuous time.** Let b be the value of leisure, suppose the future is discounted at rate: $\frac{1}{1+r}$, let the matching rate be m and denote the value of employment by V_E.

b. (4 points) Explain that the discrete time Bellman equation (or value function) for the state of unemployment is,

$$V_U = b + \frac{1}{1+r}(mV_E + (1-m)V_U)$$

c. (8 points) Derive the continuous time Bellman equation.

Question 4: Collusion and Competition Policy (16 points)

Assume that there are two identical firms in the market producing q_1 and q_2, respectively. They choose quantities simultaneously and independently and their constant symmetric marginal costs are $c_1 = c_2 = 2$. Inverse demand is given by $P = 14 - Q$, where $Q = q_1 + q_2$.

a. (6 points) Consider settings where each of the two firms can choose between either colluding and sharing monopoly profits ($\pi^M/2$, $\pi^M/2$) or competing and receiving Cournot profits. Provide matrix form representation of the simultaneous moves game described above. Identify (graphically) in the (π_1, π_2)-diagram the set of feasible payoffs and the set of sustainable payoffs in this matrix form game. (Hint: Each firm has two possible actions: collude or deviate).

b. (6 points) Next, assume that the simultaneous moves game in (a) is repeated infinitely many times. Discount factor is δ. Firms follow grim-trigger strategies. Determine graphically the set of Sub-game Perfect Nash Equilibria (SPNEa) of this infinitely repeated game. Determine the smallest value of the discount factor for which collusion on monopoly output constitutes a SPNE.

c. (4 points) Often in cartel cases lawyers, judges and economic experts state that collusion is more difficult in periods of declining demand than in periods of boom. Can you briefly describe theoretical arguments which support this view?

Question 5: Auctions (16 points)

Consider the *charity auction*. In many charity auctions, altruistic celebrities auction objects with special value for their fans to raise funds for charity. The pop-star *Madonna*, for example, held an auction to sell clothing worn during her career and raised about 3.2 million dollars. In the charity auction the winner of the lot is the highest bidder. The difference with the standard auction is that all bidders are required to pay their bid.

Suppose there are n bidders with valuations randomly drawn from the unit interval, according to the uniform distribution.

a. (12 points) Derive the equilibrium bidding function without making use of the revenue equivalence theorem.

b. (2 points) Derive the sellers expected revenue.

c. (1 points) Without making calculations. Does this auction raise more funds than the standard auctions we have seen in class (first- and second-price sealed bid auctions)? Why?

d. (1 points) Is this auction efficient from an economic point of view. If not, why? If yes, why?

Question 6: Market Design in Practice (16 points)

A teacher has a classroom with eight pupils, four boys and four girls. In the classroom there are four sets of two tables. The teacher requires that at each set of two tables one boy is sitting next to one girl. The teacher makes the following allocation of pupils over the four sets of tables:

Tables	Man	Woman
1	Adam	Kim
2	Ben	Linda
3	Chris	Mandy
4	Dan	Nicky

Of course, both the boys and the girls have a preference over who they are sitting next to. These preferences are shown in the following table:

Boys	Preferences				Girls	Preferences			
	1	2	3	4		1	2	3	4
Adam	Linda	Kim	Mandy	Nicky	Kim	Dan	Chris	Adam	Ben
Ben	Nicky	Linda	Mandy	Kim	Linda	Chris	Adam	Ben	Dan
Chris	Kim	Linda	Mandy	Nicky	Mandy	Chris	Adam	Dan	Ben
Dan	Nicky	Kim	Mandy	Linda	Nicky	Chris	Dan	Ben	Adam

All three questions below can be answered in one or two sentences. (Of course, a longer answer is allowed.)

a. (5 points) Is the allocation Pareto efficient? Explain your answer.

b. (5 points) Is the allocation stable? Explain your answer.

c. (6 points) Can the existing allocation be the result of applying deferred acceptance? If yes, explain why. If no, explain how the allocation should be modified?

Short answers of the Exam Mathematical Economics 3, March 2021

1.a. The Myerson restricted game is

$$v_L(S) = \begin{cases} 4 & \text{if } S = \{1,2,4\} \\ 8 & \text{if } S \in \{\{1,3,4\},\{1,2,3,4\}\} \\ 0 & \text{otherwise} \end{cases}$$

Harsanyi dividends of the restricted game: $\Delta_{v_L}(\{1,2,4\}) = 4$, $\Delta_{v_L}(\{1,3,4\}) = 8$ and $\Delta_{v_L}(\{1,2,3,4\}) = -4$, and thus the Myerson value is given by $\mu(v,L) = \varphi(v_L) = (\frac{4}{3} + \frac{8}{3} - \frac{4}{4}, \frac{4}{3} - \frac{4}{4}, \frac{8}{3} - \frac{4}{4}, \frac{4}{3} + \frac{8}{3} - \frac{4}{4}) = (3, \frac{1}{3}, \frac{5}{3}, 3)$.

1.b. $h^1(v,L) = (8,0,0,0)$, $h^2(v,L) = h^4(v,L) = (0,0,0,8)$, $h^3(v,L) = (0,0,4,4)$. $Core(v_L) = \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 8, 0 \leq x_1 \leq 8, 0 \leq x_2 \leq 0, 0 \leq x_3 \leq 4, 0 \leq x_4 \leq 8\}$. So, according to every Core payoff allocation, the low value buyer 2 gets 0, and the surplus of 8 is split between the seller 1, agent 4, and high value buyer 3 in such a way that buyer 3 gets at most 4 (being the difference between the two valuations). All hierarchical outcomes satisfy this property.

1.c. The marginal contribution of buyer 2 to coalition $\{1,4\}$ is $v_2 > 0$. Since all other marginal contributions of 2 are nonnegative, buyer 2 gets a positive payoff in the Myerson value. However, in every Core allocation, buyer 2 should get 0 (since the worth of 8 is already generated by coalition $\{1,3,4\}$), and thus does not belong to the Core.

2. See slides, book and exercises

3.a.

$$\underbrace{y - x_w^*}_{\text{firm accepts worker's offer } x_w^*} = \underbrace{pz + (1-p)\left(y - x_f^*\right)}_{\text{firm rejects and makes counter offer } x_f^*}$$

$$\underbrace{x_f^*}_{\text{worker accepts firm's offer } x_f^*} = \underbrace{pb + (1-p)x_w^*}_{\text{worker rejects and makes counter offer } x_w^*}$$

(i) The worker makes the first offer, x_w^* which is the highest wage that the firm will accept. The firm is residual claimant and receives output minus wage payments and it should receive at least the same payoff that it can obtain if it rejects the worker's offer. If the firm rejects the offer it risks a breakdown with probability p and it receives its outside option, z. Else, it makes the lowest wage offer x_f^* that the worker accepts and its profits are: $y - x_f^*$. (ii) the firm makes the first offer. If the worker rejects, it runs the risk of breakdown, else it makes a counter offer.

3.b. Today you receive b, the next period is discounted and two things may happen with probability m, you get matched and with probability $(1-m)$ you remain unemployed.

3.c. Multiply all parameters that depend on the unit of time by Δ.

$$\begin{aligned}
V_U &= b\Delta + \frac{1}{1+r\Delta}\left(m\Delta V_E + (1-m\Delta)V_U\right) \\
\left(1-\frac{1}{1+r\Delta}\right)V_U &= b\Delta + \frac{m\Delta}{1+r\Delta}\left(V_E - V_U\right) \\
\frac{r\Delta}{1+r\Delta}V_U &= b\Delta + \frac{m\Delta}{1+r\Delta}\left(V_E - V_U\right) \\
\frac{r}{1+r\Delta}V_U &= b + \frac{m}{1+r\Delta}\left(V_E - V_U\right) \\
\lim_{\Delta\longrightarrow 0}\frac{r}{1+r\Delta}V_U &= \lim_{\Delta\longrightarrow 0} b + \frac{m}{1+r\Delta}\left(V_E - V_U\right) \\
rV_U &= b + m\left(V_E - V_U\right)
\end{aligned}$$

4.a. Profit maximizing price and quantity in case when two firms act as a single monopolist are given by $Q^M = 6$, $P^M = 8$. Monopoly profit $\pi^M = 36$ (hence, $\pi_1 = \pi_2 = \pi^M/2 = 18$).
Standard Cournot analysis implies following best response functions: $q_1(q_2) = (12 - q_2)/2$ and $q_2(q_1) = (12 - q_1)/2$.
This implies that $q_1^* = q_2^* = 4$ and Cournot profits are given by $\pi_1^* = \pi_2^* = 16$.
Matrix form representation of the simultaneous move game described above is given in the diagram:

[natheight=1.696800in, natwidth=6.490400in, height=1.7331in, width=6.2924in]QQ32BQ01.wmf

NE of this simultaneous moves game is (Deviate, Deviate) with payoffs ($\pi_1^* = 16$, $\pi_2^* = 16$).
The set of feasible payoffs is given by the diamond area in the right hand side diagram, i.e. the convex combinations of the per-period pure strategy payoffs. The set of sustainable payoffs is given by the green area

4.b. If simultaneous moves game in (a) is repeated infinitely many times, Folk Theorem result implies that any outcome in the set of sustainable payoffs identified in (a) can arise as a SPNE of this infinitely repeated one-stage game.
The constraint for collusion on monopoly output (i.e. $q_1 = q_2 = 3$ and $\pi_1 = \pi_2 = 18$) to be sustainable as a SPNE in the industry with 2 firms is as follows $V^{collusion} \geq V^{deviation}$, or

$$\frac{18}{1-\delta} \geq 20.25 + \frac{16\delta}{1-\delta}.$$

From this inequality, we can obtain the critical discount factor for collusion to be sustainable

$$\delta \geq \frac{20.25 - 18}{20.25 - 16} = \frac{9}{17} \approx 0.5294$$

4.c. Firstly, in periods of boom future gains from collusion are relatively large compared to current gains from deviation. Also loss in the punishment phase is relatively larger. Hence, ICC is easier to satisfy.

Second argument relates to Green and Porter model. In periods of declining demand the reduction in demand (and hence lower profits) can be misinterpreted by rivals as deviations. This would destabilize cartels.

5.a.
The bidding function follows from maximising

$$u(b_i; b(v)) = (b^{-1}(b_i))^{n-1} v_i - b_i.$$

Taking the FOC gives

$$(n-1)(b^{-1}(b_i))^{n-2} \frac{db^{-1}(v_i)}{dv_i} v_i - 1 = 0$$

Applying symmetry $b_i = b(v_i)$ gives:

$$(n-1)v_i^{n-2} \frac{1}{b'(v_i)} v_i - 1 = 0,$$

or equivalently

$$(n-1)v_i^{n-1} = b'(v_i).$$

Integrating on the left and in the right gives the bidding function:

$$b(v) = \frac{n-1}{n} v^n.$$

5.b. The revenue per bidder is given by the integral $\int_0^1 \frac{n-1}{n} v^n dv = \frac{n-1}{n(n+1)}$ and from all the bidders together we get $(n-1)/(n+1)$.

5.c. The auction is *standard* because the item is allocated to the highest bidder and the payment of the bidder with the lowest valuation is zero. Therefore, the revenue equivalence theorem applies, which implies that the revenues is the same as in any other standard auction (including first-price and second-price auctions).

5.d. It is efficient because the item is always allocated to a bidder and it is allocated to the bidder who values it most.

6.a. Yes. There is no pair of boys who want to trade places, and the same holds for the girls.

6.b. No, for example Adam and Linda want to sit next to each other rather than to their current place.

6.c. No, Deferred-Acceptance provides a stable allocation. A stable allocation is: Adam Linda; Ben Mandy; Chris Kim; Dan Nicky