Vrije Universiteit Amsterdam | **July 6 2022**
Mathematical Analysis, exercises for resit exam | **First examiner: Joost Hulshof**
8.30-11.30 | **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers. You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed.

7 questions, 10 points total, your grade is $\min(T+1, 10)$, T your total score.

Question 1. ($\frac{3}{2}$ points) Some basic theory.

a) Give the definition of $x_n \to \bar{x}$ in $\mathbb{R}$.

b) Give the limit definition of continuity of $f : [0,1] \to \mathbb{R}$ in $\xi \in [0,1]$.

c) Give the definition for the uniform convergence of a sequence of functions $f_n : [0,1] \to \mathbb{R}$.

d) Formulate a theorem for interchanging limits and integrals of a sequence of functions $f_n : [0,1] \to \mathbb{R}$.

e) Give the ε, δ-condition for $f : \mathbb{R} \to \mathbb{R}$ to be differentiable in 0 with $f'(0) = 0$ if $f(0) = 1$.

f) Give the minimal conditions on $f \in C([a,b])$ which imply the existence of $\xi \in (a,b)$ with

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

Question 2. (1 point) A special case of the intermediate value theorem: let $f \in C([0,1])$ have the property that $f(0) < 0 < f(1)$. Prove that there exists $\xi \in (0,1)$ with $f(\xi) = 0$.

Hint: let ξ be the supremum of $A = \{x \in [0,1],\ f(x) < 0\}$ and consider a sequence $x_n \in A$ with $x_n \to \xi$.

Question 3. (1 point) Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : (a,b) \to \mathbb{R}$ be differentiable.

Prove that f is Lipschitz continuous on (a,b) if and only if $f' : (a,b) \to \mathbb{R}$ is bounded.

Question 4. (1 point) Exhibit a power series solution $f(x)$ of the differential equation $f''(x) = f(x)$ that satisfies $f(0) = 1$ and $f'(0) = 0$, and explain why the power series is convergent for all $x \in \mathbb{R}$.

Question 5. (2 points) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that $f(x_n) \to \infty$ for every unbounded monotone sequence x_n in $\mathbb{R}$. Prove that f has a global minimum.

Hint: formulate and use a theorem about monotone subsequences.

Question 6. Let $f : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ satisfy $f(x) = 1 + x^2 f(x)$.

a) ($\frac{1}{2}$ point) Use the triangle inequality to prove that $|f(x)| \leq \frac{4}{3}$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$.

b) (1 point) Use (a) to prove that f is differentiable in $x = 0$.

Question 7. Define the sequence of polynomials $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_0(x) = 1, \quad f_n(x) = 1 + \int_0^x f_{n-1}(s)^2 \, ds$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Thus $f_n = \Phi(f_{n-1})$ with $\Phi(f)$ defined by

$$\Phi(f)(x) = 1 + \int_0^x f(s)^2 \, ds$$

for all $x \in \mathbb{R}$. In what follows we restrict the attention to $x \in [0, \frac{1}{4}]$.

a) ($\frac{1}{2}$ point) Assume $1 \leq f(x) \leq 2$ for all $0 \leq x \leq \frac{1}{4}$. Show $1 \leq \Phi(f)(x) \leq 1 + 4x \leq 2$ for all $0 \leq x \leq \frac{1}{4}$.

b) ($\frac{1}{2}$ point) Let f and g be polynomials with $1 \leq f(x) \leq 1 + 4x$ and $1 \leq g(x) \leq 1 + 4x$ for all $0 \leq x \leq \frac{1}{4}$.

   Show that

$$|\Phi(f)(x) - \Phi(g)(x)| \leq \frac{3}{4} \max_{0 \leq x \leq \frac{1}{4}} |f(x) - g(x)|$$

   for all $x \in [0, \frac{1}{4}]$.

   Hint: $f(x)^2 - g(x)^2 = (f(x) + g(x))(f(x) - g(x))$.

c) ($\frac{1}{2}$ point) Use (a) and (b) to show that f_n is uniformly convergent on $[0, \frac{1}{4}]$.

d) ($\frac{1}{2}$ point) Let $f : [0, \frac{1}{4}] \to \mathbb{R}$ be the limit function in (c). Explain why

$$f(x) = 1 + \int_0^x f(s)^2 \, ds$$

   holds for all $x \in [0, \frac{1}{4}]$.