VU Amsterdam	Mathematical Analysis (XB_0009)
Faculty of Sciences	Solutions to Resit Exam 2022
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Solution to Problem 1.

a) We say that $x_n \to \overline{x}$ if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - \overline{x}| < \varepsilon$.

b) The function f is continuous at ξ if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in [0,1]$ with $|x - \xi| < \delta$ there holds $|f(x) - f(\xi)| < \varepsilon$.

c) We say that (f_n) converges uniformly to f on $[0,1]$ if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in [0,1]$ we have $|f_n(x) - f(x)| < \varepsilon$.

d) If (f_n) is a sequence of integrable functions converging uniformly on $[0,1]$ to $f(x) = \lim_{n\to\infty} f_n(x)$, then f is integrable and

$$\lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x = \int_0^1 \lim_{n\to\infty} f_n(x)\mathrm{d}x\,.$$

e) The function f is differentiable at $x = 0$ with $f'(0) = 0$ and $f(0) = 1$ if and only if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|x| < \delta$, then $|f(x) - 1| < \varepsilon|x|$.

f) The function f needs to be differentiable on (a,b).

Solution to Problem 2.

The set A is non-empty since $0 \in A$. It is bounded from above by 1 and therefore, there exists the supremum $\xi \in [0,1]$ of A. We have $f(\xi) \geq 0$. Indeed, if, by contradiction, $f(\xi) < 0$, then $\xi < 1$ and taking $\varepsilon := |f(\xi)|$ there is $\delta > 0$ such that $|f(x) - f(\xi)| < |f(\xi)|$ for all $x \in [0,1]$ with $|x - \xi| < \delta$. In particular, $f(x) < f(\xi) + |f(\xi)| < 0$ for all $x \in (\xi, \xi+\delta) \cap [0,1]$. Therefore, there is $x \in A$ with $x > \xi$, a contradiction. Thus, $f(\xi) \geq 0$. For all $n \in \mathbb{N}$ there is $x_n \in A$ with $\xi - \frac{1}{n} < x_n \leq \xi$ by definition of supremum. Therefore, $f(x_n) < 0$ and, by the sandwich theorem, $x_n \to s$. We have $f(\xi) = \lim_{n\to\infty} f(x_n) \leq 0$, where the equality follows from the continuity of f and the inequality from the order limit theorem. So we have proved that $f(\xi) \geq 0$ and $f(\xi) \leq 0$. Hence, $f(\xi) = 0$.

Solution to Problem 3.

Suppose that f is Lipschitz on (a,b). This means that there is $r > 0$ such that for all $x, y \in (a,b)$, $|f(x) - f(y)| \leq L|x - y|$. Thus, for all $x \in (a,b)$, we have

$$|f'(x)| = \left|\lim_{y\to x} \frac{f(y) - f(x)}{y - x}\right| = \lim_{y\to x} \left|\frac{f(y) - f(x)}{y - x}\right| \leq r\,,$$

where in the first step we used the definition of the derivative, in the second step we used that the absolute value function is continuous and in the last step the limit order theorem together with the fact that f is Lipschitz. Thus we see that f' is a bounded function. Vice versa, assume that there is $r \geq 0$ such that $|f'(x)| \leq r$ for all $x \in (a,b)$. Let $a < x < y < b$. Then, the mean-value theorem implie that there is $c \in (x,y)$ such that $|f(y) - f(x)| = |f'(c)(y - x)| = |f'(c)||y - x| \leq r|y - x|$, which means that f is Lipschitz.

Solution to Problem 4.

Not relevant for the 2023 course.

Solution to Problem 5.

Let $x_* \in \mathbb{R}$ and consider the set $A := \{x \in \mathbb{R} \; : \; f(x) \leq f(x_*)\}$. We claim that there is $M > 0$ such that $A \subseteq [-M, M]$. If not, there is a sequence $x_n \in A$ such that $x_n > n$ or there is a sequence $x_n \in A$ such that $x_n < -n$. In any case, we can assume, up to passing to a subsequence that (x_n) is monotone and unbounded. By hypothesis, we have $f(x_n) \to \infty$. On the other hand, $f(x_n) \leq f(x_*)$ for all $n \in \mathbb{N}$ since $x_n \in A$, which is a contradiction. Since f is continuous, it admits a minimum on the compact interval $[-M, M]$, i.e., there is $x_0 \in [-M, M]$ such that $f(x) \geq f(x_0)$ for all $x \in [-M, M]$. If $x \notin [-M, M]$, then $x \notin A$ and therefore, $f(x) > f(x_*)$. On the other hand, since $x_* \in A \subset [-M, M]$, we also have $f(x_*) \geq f(x_0)$ and so $f(x) > f(x_*) \geq f(x_0)$. Therefore, for all $x \in \mathbb{R}$, we have $f(x) \geq f(x_0)$.

Solution to Problem 6.

a) We have $|f(x)| \leq 1 + x^2|f(x)| \leq 1 + \frac{1}{4}|f(x)|$. Therefore, $\frac{3}{4}|f(x)| \leq 1$ and so $|f(x)| \leq \frac{4}{3}$.

b) We have $|f(x) - 1| = |xf(x)||x| \leq \frac{4}{3}|x||x|$. Given $\varepsilon > 0$, we can take $\delta := \frac{3}{4}\varepsilon$ so that for all x with $|x| < \delta$, we get $|f(x) - 1| < \frac{4}{3}\delta|x| = \varepsilon|x|$. This means that f is differentiable at $x = 0$ with derivative $f'(0) = 0$ (and $f(0) = 1$).

Notice that this exercise can also be solved by noticing that by hypothesis $f(x) = \frac{1}{1-x^2}$ and then computing

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{1}{1-x^2} - 1}{x} = \lim_{x \to 0} \frac{x}{1 - x^2} = 0 \,.$$

Solution to Problem 7.

a) We have

$$\Phi(f)(x) = 1 + \int_0^x f(s)^2 \mathrm{d}s \leq 1 + \int_0^x 2^2 \mathrm{d}s = 1 + 4x \,,$$

where we used the monotonicity of the integral and the fact that $f(s)^2 \leq 2^2 = 4$ since $1 \leq f(s) \leq 2$.

b) We have

$$\begin{aligned}
|\Phi(f)(x) - \Phi(g)(x)| = \left| \int_0^x f(s)^2 - g(s)^2 \mathrm{d}s \right| &\leq \int_0^x |f(s)^2 - g(s)^2| \mathrm{d}s \\
&= \int_0^x |f(s) - g(s)||f(s) + g(s)| \mathrm{d}s \\
&\leq \int_0^x |f - g|_{\max}(1 + 4s + 1 + 4s) \mathrm{d}s \\
&= |f - g|_{\max}(2x + 4x^2) \\
&\leq |f - g|_{\max}(2\frac{1}{4} + 4\frac{1}{4^2}) \\
&= \frac{3}{4}|f - g|_{\max} \,.
\end{aligned}$$

c) Since $1 \leq f_0(x) \leq 1+4x \leq 2$ for all $x \in [0, \frac{1}{4}]$, we also have $1 \leq f_n(x) \leq 1+4x \leq 2$ for all $x \in [0, \frac{1}{4}]$, by induction. Therefore, for all $k \in \mathbb{N}$, we can apply b) to show that

$$|f_k - f_{k-1}|_{\max} = |\Phi(f_{k-1}) - \Phi(f_{k-2})|_{\max} \leq \frac{3}{4}|f_{k-1} - f_{k-2}|_{\max} \leq \dots \leq \left(\frac{3}{4}\right)^{k-1} |f_1 - f_0|_{\max}$$

Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be chosen in such a way that $|f_1 - f_0|_{\max} \sum_{k=N+1}^{\infty} (\frac{3}{4})^k < \varepsilon$. Therefore, for all $n \geq m \geq N$, we have

$$|f_n - f_m|_{\max} \leq \left| \sum_{k=m+1}^{n} f_k - f_{k-1} \right|_{\max} \leq \sum_{k=m+1}^{n} |f_k - f_{k-1}|_{\max} \leq \sum_{k=m+1}^{n} \left(\frac{3}{4}\right)^{k-1} |f_1 - f_0|_{\max} < \varepsilon\,.$$

Thus, (f_n) is a Cauchy sequence in $C[0, \frac{1}{4}]$ with respect to the uniform metric $d_{\max}(f, g) := |f - g|_{\max}$. Since this metric space is complete, the sequence (f_n) is convergent with respect to the uniform metric to some $f \in C[0, \frac{1}{4}]$.

d) For each $x \in [0, \frac{1}{4}]$, we can take the limit in n of $f_{n+1}(x) = 1 + \int_0^x f_n(s)^2 \mathrm{d}s$ and get

$$f(x) = 1 + \int_0^x f(s)^2 \mathrm{d}s\,,$$

where on the right we use the integrable limit theorem exploiting the fact that, since (f_n) is converging uniformly to f and all f_n are bounded, then (f_n^2) is converging uniformly to f^2.