Vrije Universiteit Amsterdam | **June 6 2022**
Mathematical Analysis, full exam, answers | **First examiner: Joost Hulshof**
8.30-11.30 | **Full examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers. You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed.

Your grade will be $\min(10, 1 + T/2)$, T your total score, maximum $T = 20$, 6c is bonus.
The first 3 questions are sufficient for grade 6. Questions 4,5,6 are on the other side.
NB Each point is half a point of the grade, in de second exam each point is 1 point of the grade.

Question 1. (4 points) Some basic theory.

a) ($\frac{1}{2}$ point) Give the ε-definition of $x_n \to \bar{x}$, both in $\mathbb{R}$ and in a metric space X.

b) ($\frac{1}{2}$ point) Give the limit definition of continuity of $f : A \to \mathbb{R}$ in $\xi \in A$, $A \subset \mathbb{R}$.

c) ($\frac{1}{2}$ point) Which single condition on a sequence $x_n \in \mathbb{R}$ implies existence of a convergent subsequence?

d) ($\frac{1}{2}$ point) For $f, g \in C([a, b])$ give the definition of the uniform distance $d(f, g)$.

e) ($\frac{1}{2}$ point) Give the ε-criterion for the integrability of a bounded function $f : [0, 1] \to \mathbb{R}$.

f) ($\frac{1}{2}$ point) Give the ε, δ-definition for the uniform continuity of a function $f : [0, 1] \to \mathbb{R}$.

g) ($\frac{1}{2}$ point) Give the ε, δ-condition for $f : \mathbb{R} \to \mathbb{R}$ to be differentiable in 0 with $f'(0) = 0$ if $f(0) = 0$.

h) ($\frac{1}{2}$ point) Give the conditions on $f : [a, b] \to \mathbb{R}$ which imply the existence of $\xi \in (a, b)$ with

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

Grading/answers. NB The last 4 are the 4 items of 1 in the second exam.

a) Definition 2.6: (2.8) with $L = \bar{x}$, as in (2.10),
Definition 5.10 second statement, without the first quantor.

b) Definition 4.1.

c) Boundedness, see Theorem 3.20 (Bolzano-Weierstrass).

d) (4.1) in Definition 4.8, norm given by Definition 4.5, repeated in first paragraph of Chapter 5.

e) Definition 8.7.

f) Theorem 7.2, first sentence.

g) $\forall_{\varepsilon>0}\, \exists_{\delta>0} : 0 < |x| < \delta \implies |R(x)| < \varepsilon\, |x|$.

Question 2. Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = 1 - \frac{1}{1+n(1+x)}.$$

a) (2 points) Prove that f_n converges uniformly on $[0,1]$.

b) (2 points) Now state and use a theorem to show that $\int_0^1 f_n(x)\,dx \to 1$ as $n \to \infty$.

Grading/answers.

a) *Since* $1+x \geq 1$ *the denominator in the fraction goes to* ∞ *for all* $x \in [0,1]$ *so the limit should be* 1. *Let* $\varepsilon > 0$. *Since*

$$|f_n(x) - 1| = |\frac{1}{1+n(1+x)}| = \frac{1}{1+n(1+x)} \leq \frac{1}{1+n} \leq \frac{1}{1+N} < \varepsilon$$

for all $n \geq N$ *provided* $N \in \mathbb{N}$ *satisfies*

$$\frac{1}{1+N} < \varepsilon.$$

Such an N *exists in view of Theorem 1.5 (The Archimedean Principle). Otherwise* $\frac{1}{\varepsilon} \in \mathbb{R}$ *would be an upper bound for* $\mathbb{N}$, *and Archimedes wouldn't allow for that.*

Thus we have verified Definition 4.18 with $f(x) = 1$ *for all* $x \in [0,1] = [a,b]$.

Alternatively you may use Definition 4.9 and say that

$$|f_n - f|_{max} = \max_{0 \leq x \leq 1} |f_n(x) - f(x)| = \max_{0 \leq x \leq 1} |\frac{1}{1+n(1+x)}| = \frac{1}{1+n} \leq \frac{1}{1+N} < \varepsilon$$

holds for $n \geq N$ *provided* N *is chosen as above.*

b) *The theorem you should know and apply is Theorem 17.3, the fundamental limit theorem. It follows that* $\int_0^1 f_n \to \int_0^1 1 = 1$ *as* $n \to \infty$. *You will also get full points for the observation that* f_n *is integrable by Theorem 8.6, and that consequently (note* $b - a = 1 - 0 = 1$*)*

$$|\int_0^1 f_n - 1| = |\int_0^1 (f_n - 1)| \leq |f_n - 1|_{max} = \frac{1}{1+n} \to 0.$$

Question 3. (2 points) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x) = \frac{x}{1 + f(x)^2}$$

for all $x \in \mathbb{R}$.

a) (1 point) Verify that

$$f(x) - x = -\frac{x f(x)^2}{1 + f(x)^2}.$$

b) (1 point) Prove that f is differentiable in $x = 0$ with $f'(0) = 1$.

Hint: denote the right hand side in a) by $R(x)$ and use $|f(x)| \leq |x|$ for its numerator.

Grading/answers. NB This is 3 in the second exam. Also for 2 points.

a) *This item prepares for $f(x) = 1\,x + R(x)$ and the conclusion that $f'(0) = 1$, whereas 1g) above is about $f(x) = 0\,x + R(x)$ and the conclusion that $f'(0) = 0$. Here we thus have*

$$R(x) = f(x) - x = \frac{x}{1 + f(x)^2} - x = \frac{x - x - x f(x)^2}{1 + f(x)^2} = -\frac{x f(x)^2}{1 + f(x)^2},$$

in which we will then want

$$\frac{f(x)^2}{1 + f(x)^2} < \varepsilon.$$

b) *Observe that by a) and the equation for $f(x)$ the remainder term $R(x)$ may be estimated by*

$$|R(x)| = |f(x) - x| = \frac{|x| f(x)^2}{1 + f(x)^2} \leq \frac{|x|\, x^2}{1 + f(x)^2} \leq |x|\, x^2 \leq \varepsilon |x|,$$

provided $x^2 < \varepsilon$. So choose $\delta > 0$ with $\delta^2 = \varepsilon$ to conclude that

$$\forall_{\varepsilon>0}\, \exists_{\delta>0} : 0 < |x| < \delta \implies |R(x)| < \varepsilon\, |x|.$$

Question 4. (4 points) Let $f : (0,1) \to \mathbb{R}$ be defined by

$$f(x) = x^3 + \frac{1}{x^2} + \frac{1}{1-x}.$$

Prove that f has a minimum in $(0,1)$.

Consider a sequence $y_n \in R_f = \{f(x) : 0 < x < 1\}$ which converges to the infimum of R_f.

Grading/answers. Not all the details below are needed for full points of course. Definition 4.1 is not really needed because $f(x)$ is given by an algebraic expression

(i) The function f is defined by a formula for $f(x)$ as sum of 3 terms. Each of these terms is positive, so 0 is a lower bound for the (clearly non-empty) range R_f of f. Let m be the largest lower bound (infimum) of R_f. Then $m \geq 0$, and for all $n \in \mathbb{N}$ the number $m + \frac{1}{n}$ is not a lower bound, implying the existence of $y_n \in R_f$ with $m \leq y_n < m + \frac{1}{n}$. In particular we now have a sequence $y_n \in R_f$ and clearly $y_n \to m$ as $n \to \infty$.

(ii) Since $y_n \in R_f$ we have by definition of R_f the existence of $x_n \in (0,1)$ with $f(x_n) = y_n$, this for every $n \in \mathbb{N}$. The sequence x_n cannot so easily been shown to converge, but 1c above tells you it has a convergent subsequence. Subsequences are denoted by x_{n_k} in Section 3.3, so let $\bar{x} = \lim_{k\to\infty} x_{n_k}$ be the limit of the subsequence, which we called limit point of the original sequence in Definition 3.21.

(iii) Since $(0,1) \subset [0,1]$ and the closed interval $[0,1]$ is closed, it follows that $\bar{x} \in [0,1]$. To complete the proof you have to show $\bar{x} \in (0,1)$. By construction $f(x_{n_k}) = y_{n_k} \to m$ as $k \to \infty$, and in particular the sequence $f(x_{n_k})$ is bounded from above by some $M > 0$. The lower estimates

$$f(x) > \frac{1}{x^2} > \frac{1}{x}, \quad f(x) > \frac{1}{1-x}$$

imply for all $k \in \mathbb{N}$ that

$$\frac{1}{x_{n_k}} < M, \quad \frac{1}{1-x_{n_k}} < M,$$

whence

$$x_{n_k} > \frac{1}{M}, \quad 1 - x_{n_k} > \frac{1}{M},$$

prohibiting $x_{n_k} \to 0$ and $x_{n_k} \to 1$.

(iv) Thus $\bar{x} \in (0,1)$ and the limit theorems (Theorem 2.36 and Theorem 2.40) now imply that $f(x_{n_k}) \to f(\bar{x})$ as $k \to \infty$. Since $f(x_{n_k}) \to m$ it follows that

$$f(\bar{x}) = m = \inf_{0<x<1} f(x),$$

whence $f(x) \geq f(\bar{x})$ for all $x \in (0,1)$. This completes the proof.

Question 5. (2 points) Suppose that the power series

$$J(x) = a_0 - a_2x^2 + a_4x^4 - a_6x^6 + \cdots = \sum_{k=0}^{\infty}(-1)^k a_{2k}x^{2k}$$

has a positive radius R of convergence.

a) (1 point) Fix $r \in (0, R)$. Suppose that J is a solution of the differential equation

$$J''(x) + \frac{1}{x}J'(x) + J(x) = 0$$

on $(0, r)$. Show that the coefficients satisfy the recurrence relation $a_{2(k-1)} = (2k)^2 a_{2k}$ for all $k \in \mathbb{N}$. Hint: use

$$J''(x) + \frac{1}{x}J'(x) = \frac{1}{x}(xJ'(x))'$$

to make the term by term calculations easier.

For the solution with $a_0 = 1$ it follows that

$$J(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k (k!)^2} x^{2k}.$$

b) (1 point) Explain why $R = \infty$.

Grading/answers. NB This 4 in the second exam.

a) *Put a few more terms and write*

$$J(x) = a_0 - a_2x^2 + a_4x^4 - a_6x^6 + a_8x^8 - a_{10}x^{10} + \cdots,$$

to conclude for $|x| < R$ that

$$J'(x) = -2a_2x + 4a_4x^3 - 6a_6x^5 + 8a_8x^7 - 10a_{10}x^9 + \cdots,$$
$$xJ'(x) = -2a_2x^2 + 4a_4x^4 - 6a_6x^6 + 8a_8x^8 - 10a_{10}x^{10} + \cdots,$$
$$(xJ'(x))' = -2^2a_2x + 4^2a_4x^3 - 6^2a_6x^5 + 8^2a_8x^7 - 10^2a_{10}x^9 + \cdots,$$
$$\frac{1}{x}(xJ'(x))' = -2^2a_2 + 4^2a_4x^2 - 6^2a_6x^4 + 8^2a_8x^6 - 10^2a_{10}x^8 + \cdots,$$

and put this equal to

$$-J(x) = -a_0 + a_2x^2 - a_4x^4 + a_6x^6 - a_8x^8 + a_{10}x^{10} - \cdots,$$

to conclude that

$$-2^2a_2 = -a_0, \quad 4^2a_4 = a_2, \quad -6^2a_6 = -a_4, \quad 8^2a_8 = a_6,$$

and so on. We recognise $(2k)^2a_{2k} = a_{2(k-1)}$ for $k = 1, 2, 3, \ldots$.
Alternatively we have for $|x| < R$ that

$$J(x) = \sum_{k=0}^{\infty}(-1)^k a_{2k}x^{2k} \implies J'(x) = \sum_{k=0}^{\infty}(-1)^k 2ka_{2k}x^{2k-1} = \sum_{k=1}^{\infty}(-1)^k 2ka_{2k}x^{2k-1}$$

$$\implies xJ'(x) = \sum_{k=1}^{\infty}(-1)^k 2ka_{2k}x^{2k} \implies (xJ'(x))' = \sum_{k=1}^{\infty}(-1)^k(2k)^2a_{2k}x^{2k-1}$$

$$\implies \frac{1}{x}(xJ'(x))' = \sum_{k=1}^{\infty}(-1)^k(2k)^2a_{2k}x^{2k-2},$$

which equated to

$$-J(x) = -\sum_{k=0}^{\infty}(-1)^k a_{2k}x^{2k} = \sum_{k=1}^{\infty}(-1)^k a_{2k-2}x^{2k-2}$$

gives $a_{2k-2} = (2k)^2 a_{2k}$.

Without the hint you would do

$$\frac{J'(x)}{x} = -2a_2 + 4a_4x^2 - 6a_6x^4 + 8a_8x^6 - 10a_{10}x^8 + \cdots,$$

$$J''(x) = -2a_2 + 4\cdot 3a_4x^2 - 6\cdot 5a_6x^4 + 8\cdot 7a_8x^6 - 10\cdot 9a_{10}x^8 + \cdots,$$

whence

$$J''(x)+\frac{1}{x}J'(x)+J(x) = a_0-2a_2-2a_2+(-a_2+4a_4+4\cdot 3a_4)x^2+(a_4-6a_6-6\cdot 5a_6)x^4+(-a_6+8a_8+8\cdot 7a_8)x^6+\cdots,$$

which is zero if $a_0 = (2+2)a_2 = 2\cdot 2a_2$, $a_2 = 4\cdot 4a_4$, $a_4 = 6\cdot 6a_6$, *and so on. You should recognise* $a_{2(k-1)} = (2k)^2 a_{2k}$ *for all* $k \in \mathbb{N}$.

Alternatively we have for $|x| < R$ *that*

$$J(x) = \sum_{k=0}^{\infty}(-1)^k a_{2k}x^{2k} \implies J'(x) = \sum_{k=0}^{\infty}(-1)^k 2k a_{2k}x^{2k-1} = \sum_{k=1}^{\infty}(-1)^k 2k a_{2k}x^{2k-1} \implies$$

$$\frac{J'(x)}{x} = \sum_{k=1}^{\infty}(-1)^k 2k a_{2k}x^{2k-2}, \quad J''(x) = \sum_{k=0}^{\infty}(-1)^k 2k(2k-1)a_{2k}x^{2k-2},$$

whence

$$J''(x) + \frac{J'(x)}{x} = \sum_{k=1}^{\infty}(-1)^k(2k(2k-1)+2k)a_{2k}x^{2k-2} = \sum_{k=1}^{\infty}(-1)^k(2k)^2 a_{2k}x^{2k-2},$$

which equated to

$$-J(x) = -\sum_{k=0}^{\infty}(-1)^k a_{2k}x^{2k} = \sum_{k=1}^{\infty}(-1)^k a_{2k-2}x^{2k-2}$$

gives $a_{2k-2} = (2k)^2 a_{2k}$.

b) The recurrence $a_{2(k-1)} = (2k)^2 a_{2k}$ *implies the formula for* $J(x)$, *with the individual terms going to zero for each* x. *As discussed in the last week this implies that* $R \geq |x|$ *for every* x *so* $R = \infty$.

Question 6. (4 points) Let $x \in [0,1]$ and let $F_x : [0,\infty) \to \mathbb{R}$ be defined by

$$F_x(y) = 1 + x + \frac{y}{1+y}.$$

a) (1 point) Show that F_x is a contraction with contraction factor $\frac{1}{4}$ from $[1,\infty)$ to itself.

For every function $f : [0,1] \to [1,\infty)$ we define a new function $g = \Phi(f)$ by setting $g(x) = F_x(f(x))$ for all $x \in [0,1]$. This defines a map Φ from

$$A = \{f \in C([0,1]) : \forall_{x \in [0,1]}\ f(x) \geq 1\}$$

to itself. You don't have to prove that $g = \Phi(f)$ is continuous for every $f \in A$, and you may use that A is a closed subset of $C([0,1])$.

b) (1 point) Prove that Φ has a unique fixed point.

Bonus. Define Ψ by setting

$$g = \Psi(f), \quad g(x) = \int_0^x F_s(f(s))\,ds = \int_0^x (1 + s + \frac{f(s)}{1+f(s)})\,ds$$

for every f in

$$B = \{f \in C([0,1]) : \forall_{x \in [0,1]}\ f(x) \geq x\}.$$

You don't have to prove that B is a closed subset of $C([0,1])$.

c) (2 points) Prove that Ψ has a unique fixed point in B.

Hint: modify the estimate in a) using $f(s) \geq s$ to derive that

$$|F_s(f_1(s)) - F_s(f_2(s))| \leq \frac{|f_1 - f_2|_{max}}{(1+s)^2}$$

Grading/answers. NB 6c is more less 5 in the second exam. In 5b you have to remark that Ψ maps B to B. In 5c you have to remar that every nonnegative solution is in B.

Note that Φ in 6 of the full exam maps

$$X = \{f \in C([0,1]) : \forall_{x \in [0,1]}\ f(x) \geq 0\}$$

to A, and likewise Ψ maps X to B. So the solutions of $\Phi(f) = f$ and $\Psi(f) = f$ are unique in X.

a) *Fix $x \in [0,1]$. Clearly $F_x(y) = 1 + x + \frac{y}{1+y} \geq 1$ for every $y \in [1,\infty)$, so F_x is a map from $[1,\infty)$ to itself. We only used $x \geq 0$ to conclude so. What follows holds for all x.*

Since

$$F_x(y_1) - F_x(y_2) = \frac{y_1}{1+y_1} - \frac{y_2}{1+y_2} = \frac{(1+y_2)y_1 - (1+y_1)y_2}{(1+y_1)(1+y_2)} = \frac{y_1 - y_2}{(1+y_1)(1+y_2)}$$

implies

$$|F_x(y_1) - F_x(y_2)| = |\frac{y_1 - y_2}{(1+y_1)(1+y_2)}| = \frac{|y_1 - y_2|}{(1+y_1)(1+y_2)} \leq \frac{|y_1 - y_2|}{(1+1)(1+1)} = \frac{1}{4}|y_1 - y_2|$$

for all $y_1, y_2 \in [1,\infty)$, the map F_x is contractive on $[1,\infty)$ with contraction factor $\frac{1}{4}$ for every x.

b) *Now let f_1 and f_2 be continuous functions from the x-interval to $[0,\infty)$, and let $g_1 = \Phi(f_1)$, $g_2 = \Phi(f_2)$. Then by a) it follows that*

$$|g_1(x) - g_2(x)| \leq \frac{1}{4}|f_1(x) - f_2(x)| \leq \frac{1}{4}|f_1 - f_2|_{max}$$

for every $x \in [0,1]$, since f_1 and f_2 have $f_1(x) \geq 1$ and $f_2(x) \geq 1$ for all $x \in [0,1]$. Here we use the maximum norm on $C([0,1])$. Since g_1 and g_1 are continuous this implies that the maximum of $|g_1(x) - g_2(x)|$ on $[0,1]$ exists and is less or equal than the right hand side, that is

$$|\Phi(f_1) - \Phi(f_2)|_{max} = |g_1 - g_2|_{max} \leq \frac{1}{4}|f_1 - f_2|_{max}.$$

This holds for all f_1, f_2 in

$$A = \{f \in C([0,1]) : \forall_{x\in[0,1]}\ f(x) \geq 1\}.$$

So $\Phi : A \to A$ is contraction on the closed subset A of the complete metric space $C([0,1])$. It follows that Φ has a unique fixpoint.

c) *Since*

$$1 \leq (1 + s + \frac{f(s)}{1+f(s)}) \leq 1 + 1 + 1 = 3$$

every such function g has

$$g(x) = \int_0^x (1 + s + \frac{f(s)}{1+f(s)})\, ds \geq \int_0^x 1 = x$$

for every x, and is Lipschitz continuous with Lipschitz constant 3. It follows that Ψ maps B to B. Following the hint

$$|F_x(y_1) - F_x(y_2)| = |\frac{y_1 - y_2}{(1+y_1)(1+y_2)}| = \frac{|y_1 - y_2|}{(1+y_1)(1+y_2)} \leq \frac{|y_1 - y_2|}{(1+x)(1+x)} = \frac{1}{(1+x)^2}|y_1 - y_2|,$$

so

$$\begin{aligned} |g_1(x) - g_2(x)| &= |\int_0^x F_s(f_1(s))\, ds - \int_0^x F_s(f_2(s))\, ds| = |\int_0^x (F_s(f_1(s)) - F_s(f_2(s))\, ds| \\ &\leq \int_0^x |F_s(f_1(s)) - F_s(f_2(s))|\, ds \leq \int_0^x \frac{1}{(1+s)^2}|f_1(s) - f_2(s))|\, ds \\ &\leq \int_0^x \frac{|f_1 - f_2|_{max}}{(1+s)^2}\, ds = \int_0^x \frac{1}{(1+s)^2}\, ds\, |f_1 - f_2|_{max} \leq \frac{1}{2}|f_1 - f_2|_{max}. \end{aligned}$$

Thus $\Psi : B \to B$ is a contraction and thereby has a unique fixpoint.