Vrije Universiteit Amsterdam	**Fool's Day 2022**
Mathematical Analysis, midterm exam	**First examiner: Joost Hulshof**
15.30-17.45	**Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers. You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed.

Your grade will be $1 + T$, T your total score, with a maxium of 10.
The first 3 exercises are sufficient to pass the midterm, 4 and 5 are on the other side.

Problem 1. Some basic theory needed below.

a) ($\frac{1}{2}$ point) Give the definition of a convergent sequence in $\mathbb{R}$.

Answer: the sequence x_n is convergent if

$$\exists_{\bar{x}\in\mathbb{R}}\,\forall_{\varepsilon>0}\,\exists_{N\in\mathbb{N}}\,\forall_{n\geq N}\;:\;\underbrace{|x_n-\bar{x}|}_{d(x_n,\bar{x})}<\varepsilon.$$

You really need to mention the limit!

b) ($\frac{1}{2}$ point) What do you need to assume on a subset of $\mathbb{R}$ to know that the infimum of that set exists?

Answer: the subset needs to be non-empty and bounded from below.

NB1 Sets are not monotone. But given a non-empty subset A of $\mathbb{R}$ and a lower bound in $\mathbb{R}$ of A, $m = \inf A$ exists, and thereby a sequence $x_n \in A$ with $m \leq x_n \to m$ as $n \to \infty$.

NB2 If you include the assumption that A is closed then the statement is of no use for analysis because we apply it to non-closed sets, as well as to prove that sets are closed.

c) ($\frac{1}{2}$ point) Formulate the Banach Fixed Point Theorem for contractions on closed subsets A of $\mathbb{R}$.

Answer: Let A be closed and $f : A \to A$ be a contraction. That is, f maps A to A and

$$\exists_{\theta\in[0,1)}\forall_{x,y\in A} : |f(x)-f(y)| \leq \theta|x-y|.$$

Then there exists a unique solution $x = \bar{x}$ of $x = f(x)$ in A. Every sequence with $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$ and $x_0 \in A$ converges to $\bar{x}$.

NB The uniqueness of the solution of $x = f(x)$ in A is an essential part of the statement, the statement about the sequences elaborates on the proof, and is not needed in most applications.

Problem 2. Let the sequence x_n indexed by $n \in \mathbb{N}$ be defined by

$$x_n = \frac{n}{n^2 - 2} \quad \text{for} \quad n = 1, 2, 3, \dots,$$

and note that $x_n > \frac{1}{n}$ for all $n \in \mathbb{N}$ with $n \geq 2$.

NB Not for $n = 1$, my bad. This was to prevent you from using $x_n \leq \frac{1}{n}$.

In this exercise you should use that no positive number is an upper bound for the set $\mathbb{N}$.

a) ($\frac{1}{2}$ point) Show that $x_n \leq \frac{2}{n}$ for all $n \geq 2$.

Answer: to check the inequality note that when $n^2 > 2$ these equivalences hold:

$$\frac{n}{n^2 - 2} \leq \frac{2}{n} \iff n^2 \leq 2(n^2 - 2) \iff n^2 \geq 4$$

So for $n \geq 2$ the inequality follows.

NB1 This was meant to be easy and help you in b). Don't even think about using induction here.

NB2 If you do you need to prove that $x_{n+1} \leq \frac{2}{n+1}$ (don't forget the $+1$), assuming $x_n \leq \frac{2}{n}$, starting from $n = 2$ for which $x_2 \leq \frac{2}{2}$ is clear. You can then prove $x_{n+1} \leq \frac{2}{n+1}$, but really: the induction assumption is unlikely to be of much help then.

b) (1 point) Prove that x_n is convergent. Hint: guess the limit first.

Answer: your guess should be zero. For the proof you have to estimate

$$|x_n - 0| = \frac{n}{n^2 - 2}.$$

Without the hint in (a) you should note that you cannot estimate this by $\frac{1}{n}$. Splitting n^2 in two parts

$$\frac{n}{n^2 - 2} = \frac{n}{\frac{n^2}{2} + \frac{n^2}{2} - 2} \leq \frac{n}{\frac{n^2}{2}} = \frac{2}{n}$$

for $n \geq 2$ gives the same estimate, so you find that $|x_n - 0| \leq \frac{2}{n}$ for $n \geq 2$.

With the information in (a) you can start directly with let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Then

$$|x_n - 0| \leq \frac{2}{n} \leq \frac{2}{N} = 2\,\frac{1}{N} < 2\,\frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq N$. This completes the proof.

Alternatively you can argue that you need to get $\frac{2}{n}$ smaller than ε. If this is not possible than $\frac{2}{n} \geq \varepsilon$ for all $n \in \mathbb{N}$, whence $n \leq \frac{2}{\varepsilon}$ for all n. But then $\mathbb{N}$ would be bounded in $\mathbb{R}$ from above, a contradiction. So there must exist $N \in \mathbb{N}$ with $\frac{2}{N} < \varepsilon$. And then also $\frac{2}{n} < \varepsilon$ for all $n \geq N$.

Problem 3. Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by

$$f_n(x) = 1 - \frac{1}{1 + nx}$$

a) ($\frac{1}{2}$ point) Determine the limit of the sequence $f_n(x)$ for every $x \in [0, \infty)$. No proof needed.

Answer: For $x = 0$ we have $f_n(0) = 0$, which does not go anywhere, its limit is 0. For $x > 0$ your calculus experience tells you that the second terms goes to 0, and then the limit of the whole expression should be 1. You will prove it with (c) below.

NB If you fail to note that the pointwise limit has $f(0) = 0$ and $f(x) = 1$ for $x > 0$, and then you can give a direct proof that f_n does not converge uniformly to this (wrong) f. And discover that you had the wrong limit function....

b) (1 point) Is the sequence f_n uniformly convergent on $[0, 1]$?

Give a direct proof of your answer or invoke a theorem we proved.

Answer: the limit in (a) defines a function f with $f(0) = 0$ and $f(x) = 1$ for $x > 0$. This function is discontinuous in 0. But we have a theorem that says that the uniform limit of a sequence of continuous functions must be continuous. Thus the convergence cannot be uniform.

NB1 It is wrong to say that the convergence is not uniform because there are two limits of $f_n(x)$: uniform convergence of a sequence of continuous functions does not imply that the limit function only takes one value!

NB2 The Banach Fixed Point Theorem is of no use here.

Alternatively: exhibit $\varepsilon > 0$ for which $|f_n(x) - f(x)| \geq \varepsilon$ for some $x \in [0, 1]$, probably depending on n, for arbitrarily large n. This x must be positive in view of $f_n(0) = 0$ for all n. Evaluate

$$|f_n(x) - f(x)| = \frac{1}{1 + nx}$$

and put $x = \frac{1}{n}$ to find

$$|f_n(\frac{1}{n}) - f(\frac{1}{n})| = \frac{1}{1 + 1} = \frac{1}{2}.$$

Let $\varepsilon = \frac{1}{2}$. For all n it is now impossible to have $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0, 1]$. So the convergence is not uniform.

NB1 With the right pointwise limit function f you cannot use the maximum norm of $f_n - f$ on $[0, 1]$ because f is not continuous on $[0, 1]$ and neither is $f_n - f$!

NB2 It is true that $f_n \to f$ uniformly, $f_n \in C([0, 1])$, together with $[0, 1] \ni x_n \to 0$ implies that $f_n(x_n) \to f(0)$, and with $x_n = \frac{1}{n}$ this gives a contradiction.

NB3 Don't use an $M = \frac{1}{x}$-trick. It may mislead you to conclude to uniform convergence!

c) (1 point) Let $a > 0$. Prove that f_n is uniformly convergent on $[a, \infty)$.

Answer: to get rid of of x in $|f_n(x) - f(x)|$ estimate

$$|f_n(x) - f(x)| = \frac{1}{1+nx} \leq \frac{1}{1+na}$$

and let $\varepsilon > 0$. The argument is now very much like in 2(b). Choose $N \in \mathbb{N}$ with $\frac{1}{1+Na} < \varepsilon$. This is possible for otherwise $\mathbb{N}$ would be bounded from above by $\frac{1}{a}(\frac{1}{\varepsilon} - 1))$ in $\mathbb{R}$. For all $n \geq N$ and all $x \geq a$ it now follows that

$$|f_n(x) - f(x)| = \frac{1}{1+nx} \leq \frac{1}{1+na} < \varepsilon.$$

This completes the proof.

NB1 There is no reason to get rid of the 1 in $1 + nx$ or $1 + na$. It does not help, but it is not wrong, you can handle $\frac{1}{na}$ just as easily: $|f_n(x) - f(x)| = \frac{1}{1+nx} \leq \frac{1}{1+na} \leq \frac{1}{na} \leq \frac{1}{Na} < \varepsilon$ by taking $N > \frac{1}{a\varepsilon}$.

NB2 This last step is what we might have called a $M = \frac{1}{a}$-trick in the notes. Choosing $N > \frac{1}{\varepsilon}$ you end up with $|f_n(x) - f(x)| < \frac{\varepsilon}{a}$. To be straight in the teachings you were told to introduce $\tilde{\varepsilon} = \frac{\varepsilon}{M} = a\varepsilon$ in the ε-statement you have at your disposal, which is the Archimedes statement in this case. Don't use an $M = \frac{1}{x}$-trick when it's about uniform convergence.

Problem 4. Let $f : (0,1) \to \mathbb{R}$ be defined by

$$f(x) = x + \frac{1}{x} + \frac{1}{1-x},$$

and let

$$R_f = \{f(x) : x \in (0,1)\}$$

be the range of f.

a) ($\frac{1}{2}$ point) Explain why the infimum $m = \inf R_f$ exists as a nonnegative number.

   Answer: use 1(b) and the fact that R_f is bounded from below by 0.

   So the largest lower bound m must have $m \geq 0$.

   All this has nothing to do with bounds on x!

b) (1 point) Prove that there exists a monotone sequence x_n in $(0,1)$ such that $f(x_n) \to m$ as $n \to \infty$. Hint: use the monotone subsequence theorem.

   Answer: to invoke the theorem you need a sequence. Since m is the infimum of R_f there exists for every $n \in \mathbb{N}$ a number $y_n \in R_f$ such that $m \leq y_n < m + \frac{1}{n}$. The fact that $y_n \in R_f$ means there exists $x_n \in (0,1)$ such that $f(x_n) = y_n$. Apply the monotone subsequence theorem to this sequence to obtain

$$n_1 < n_2 < n_3 < n_4 < \cdots$$

   such that x_{n_k} is monotone in k.

   NB All this has nothing to do with monotonicity or boundedness of the sequence $f(x_n) = y_n$!

c) (1 point) Prove that $m > 0$. Hint: show that the limit of the monotone sequence in (ii) is in $(0,1)$.

   NB. This is not the same question as (a)! Answer: Since x_{n_k} is bounded and monotone it follows that its limit $\bar{x}$ exists, either as supremum, or as infimum of the subsequence. In case $\bar{x} \in (0,1)$ it follows from the limit theorems that

$$f(x_{n_k}) = x_{n_k} + \frac{1}{x_{n_k}} + \frac{1}{1 - x_{n_k}} \to \bar{x} + \frac{1}{\bar{x}} + \frac{1}{1-\bar{x}} = f(\bar{x}) > 0,$$

   but since x_n was chosen such that $m \leq f(x_n) < m + \frac{1}{n}$ we also have $f(x_{n_k}) \to m$. It follows that $m = f(\bar{x}) > 0$.

   It remains to exclude $\bar{x} = 0$ and $\bar{x} = 1$. To do so we use the theorem that convergent sequences are bounded. For $\bar{x} = 1$ use that $f(x_{n_k}) > \frac{1}{1-x_{n_k}}$ to conclude that $f(x_{n_k})$ would be unbounded, contradicting its convergence to m. For $\bar{x} = 0$ use that $f(x_{n_k}) > \frac{1}{x_{n_k}}$ to conclude that $f(x_{n_k})$ would be unbounded, contradicting again its convergence to m. This completes the proof.

   NB1 Note that the continuity of f is not explicitly needed, just the limit theorems for sequences applied to the subsequence.

   NB2 A quicker proof: show directly that $f(x) = x + \frac{1}{x} + \frac{1}{1-x} > \frac{1}{x} + \frac{1}{1-x}$, whence $f(x) > \frac{1}{x} \geq \frac{1}{2}$ for $x \leq \frac{1}{2}$ and likewise $f(x) > \frac{1}{1-x} \geq \frac{1}{2}$ for $x \geq \frac{1}{2}$. Or with some more care that $f(x) = x + \frac{1}{x} + \frac{1}{1-x} > \frac{1}{x} + \frac{1}{1-x} \geq 1$.

NB Noting that every x_n depends on a this next problem easily reformulates as an exercise about a sequence of functions, say g_n, of a, viewing the parameter a as a variable. Renaming a by t perhaps and writing $g_n(t) = x_n$, the functions g_n may be considered as lying in $C([a, b])$.

Problem 5. Let $a > 1$ and let $f_a : [0, \infty) \to \mathbb{R}$ be defined by

$$f_a(x) = a - \frac{1}{x+1}.$$

Define the sequence x_n by $x_0 = 0$ and $x_n = f_a(x_{n-1})$ for all $n \in \mathbb{N}$.
Choose $A > 0$ such that $f_a(0) = a - 1 \geq A$.

a) (1 point) Show that f_a is a contraction from $[A, \infty)$ to itself.

Answer: Note that A can be any number in the (non-empty) interval $(0, a-1)$. Is $f_a(x) \geq A$ when $x \geq A$? We have

$$f_a(x) = a - \frac{1}{x+1} \geq f_a(x) = a - \frac{1}{A+1} \geq 1 + A - \frac{1}{A+1} \geq A$$

because $a \geq A + 1$ and $A > 0$. Yes, indeed f_a maps $[A, \infty)$ to itself.

Is f_a contractive on $[A, \infty)$? We have

$$|f_a(x) - f_a(y)| = \left| a - \frac{1}{x+1} - a + \frac{1}{y+1} \right| = \left| \frac{1 + x - 1 - y}{(1+x)(1+y)} \right| = \frac{|x-y|}{(1+x)(1+y)} \leq \frac{|x-y|}{(1+A)^2}$$

for all $x, y \geq A$, so f_a is contractive with contraction factor $\theta = \frac{1}{(1+A)^2} < 1$.

The last step in the chain is essential. You cannot have x or y in the expression for θ.

b) ($\frac{1}{2}$ point) Prove that x_n converges to a positive limit $\bar{x}$ as $n \to \infty$, and that

$$\bar{x} + \frac{1}{1+\bar{x}} = a.$$

Answer: note that $[A, \infty)$ is closed[1] But you cannot directly apply 1(c) because $x_0 = 0 \notin [A, \infty)$. However from (a) you know that $x_1 = f_a(0) \geq A$, so starting the iteration from x_1 the sequence x_n converges to a fixed point $\bar{x}$ of f_a in A. Re-arrange $\bar{x} = f_a(\bar{x})$ to find the equation for $\bar{x}$ in the exercise.

c) (bonus point) Prove or disprove that x_n is a monotone sequence.

Answer: this one of these cases where proof by induction (domino principle) does indeed help to find the answer. We know that $x_1 > x_0$ since $x_1 \geq A > 0 = x_0$. Now suppose that we know that $x_n > x_{n-1}$ for some $n \in \mathbb{N}$. Then

$$x_{n+1} - x_n = f_a(x_n) - f_a(x_{n-1}) = a - \frac{1}{x_n+1} - a + \frac{1}{x_{n-1}+1} = \frac{x_n - x_{n-1}}{(1+x_n)(1+x_{n-1})} > 0$$

so $x_{n+1} > x_n$. Apply this with $n = 1$ to find $x_2 > x_1$, with $n = 2$ to find $x_3 > x_2$, and so on. Since clearly $A < x_n < a$ for all $n > 1$ it follows that the limit of the sequence exists as the supremum of the sequence in $(A, a]$.

NB So this could be formulated with induction, if you like...

[1] 'closed' defined using limit statements. The definition via openness of the complement in $\mathbb{R}$ is of no direct use here.