Vrije Universiteit Amsterdam **June 29 2021**
Mathematical Analysis, full resit exam **First examiner: Joost Hulshof**
3 hours **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers. You can and *will have to use earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed.

Your grade will be $1+T$, T your total score, maximal $T=9$.

Problem 1. (2 points) Some basic theory needed for this exam.

a) ($\frac{1}{3}$ point) Give an ε, N-definition for the sequence $f_n \in C([0,1])$ to be a Cauchy sequence in $C([0,1])$.

b) ($\frac{1}{3}$ point) Let $f : [0,1] \to \mathbb{R}$ be a bounded function. For a partition P given by $N \in \mathbb{N}$ and points

$$0 = x_0 \leq x_1 \leq \cdots \leq x_N = 1$$

we write

$$\sum_{n=1}^{N} m_n(x_n - x_{n-1}) \,=\, \underline{S} \,\leq\, \bar{S} \,=\, \sum_{n=1}^{N} M_n(x_n - x_{n-1})$$

in which

$$m_n = \inf_{x\in[x_{n-1},x_n]} f(x), \quad M_n = \sup_{x\in[x_{n-1},x_n]} f(x).$$

Formulate an ε-statement that characterises the integrability of f on $[0,1]$ in terms of differences $\bar{S} - \underline{S}$.

c) ($\frac{1}{3}$ point) Formulate the Banach Fixed Point Theorem for maps $f:\,[0,1] \to [0,1]$.

d) ($\frac{1}{3}$ point) Formulate the Banach Fixed Point Theorem for maps $\Phi:\,C([0,1]) \to C([0,1])$.

e) ($\frac{1}{3}$ point) Let $f:\,\mathbb{R} \to \mathbb{R}$ and let $R(x) = f(x) - x$ for $x \in \mathbb{R}$.
Give the ε, δ-statement on the remainder term $R(x)$ for f to be differentiable in $x=0$ with $f'(0)=1$.

f) ($\frac{1}{3}$ point) Let (a,b) be an open non-empty interval in $\mathbb{R}$, and let $f \in C([a,b])$ be differentiable on (a,b).
Formulate the Mean Value Theorem for f on $[a,b]$.

Problem 2. (1 point) Prove that $2x = 1 + \sin x$ has a unique solution in $[0,1]$.
You may use what you know about cos and sin from calculus.

Problem 3. (1 point) Let $f:\,[0,1] \to \mathbb{R}$ and assume that $|f(x) - f(y)| \leq |x-y|$ for all $x,y \in [0,1]$.
Prove that f is integrable on $[0,1]$.

Problem 4. (1 point) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{1 + x^{\frac{4}{5}} + x^4}{1 + x^4}$$

for all $x \in \mathbb{R}$.
Use the ε, δ-statement for the appropriate remainder term to show that f is differentiable in $x=0$.

With everything correct so far your grade will be at least 6.

Recall that $C([0,1])$, the space of all real valued continuous functions on $[0,1]$, is complete with respect to the metric d defined by the maximum norm:

$$d(f,g) = \max_{0\leq x\leq 1} |f(x)-g(x)| = |f-g|_{max}, \quad f,g\in C([0,1]).$$

Problem 5. ($\frac{3}{2}$ points) Define $\Phi : C([0,1]) \to C([0,1])$ by

$$\Phi(f)(x) = 3 + \int_0^x \frac{f(s)}{2+s}\,ds$$

for all $x\in[0,1]$ and all $f\in C([0,1])$. You don't have to prove that Φ is well defined.
Prove that Φ has a unique fixed point in $C([0,1])$.

Problem 6. Let $p(x)$ be a power series of the form

$$p(x) = x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots,$$

in which the coefficients a_{2n+1} indexed by $n\in\mathbb{N}$ are all positive.

a) ($\frac{1}{2}$ point) Find an expression for a_{2n+1}, $n\in\mathbb{N}$, if it is given that

$$p''(x) = p(x)$$

for every $x\in[0,1]$.

Write f_n for the function defined by

$$f_n(x) = x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots + a_{2n+1}x^{2n+1} = x + \sum_{k=1}^{n} a_{2k+1}x^{2k+1}$$

for all $x\in[0,1]$.

b. ($\frac{1}{2}$ point) Show that $f_n(1)$ is a convergent sequence in $\mathbb{R}$.

c. ($\frac{1}{2}$ point) Show that f_n is a convergent sequence in $C([0,1])$.

Problem 7. (1 point) Let $f_n\in C([0,1])$ be a bounded sequence with the property that $f_n(q)$ is a Cauchy sequence in $\mathbb{R}$ for every $q\in\mathbb{Q}\cap[0,1]$.

a. Let $\varepsilon>0$ and $\delta>0$ and assume that

$$\forall_{x,y\in[0,1]}\,\forall_{n\in\mathbb{N}} : |x-y|<\delta \implies |f_n(x)-f_n(y)|<\varepsilon$$

Prove that there is an $N\in\mathbb{N}$ such that $|f_n-f_m|_{max}<3\varepsilon$ for all $m,n\geq\mathbb{N}$.

Hint: use

$$|f_n(x)-f_m(x)| \leq |f_n(x)-f_n(q)| + |f_n(q)-f_m(q)| + |f_m(q)-f_m(x)|$$

with

$$q = \frac{j}{N},$$

$j\in\{0,1,\ldots,N\}$ and N fixed. How would Archimedes choose N?

b. Assume that the ε,δ-statement

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{x,y\in[0,1]}\,\forall_{n\in\mathbb{N}} : |x-y|<\delta \implies |f_n(x)-f_n(y)|<\varepsilon$$

holds. Use part a to prove that f_n is a Cauchy sequence in $C([0,1])$.