VU Amsterdam	Mathematical Analysis (XB_0009)
Faculty of Sciences	Solutions to Resit Exam 2021
Gabriele Benedetti	

Solution to Problem 1.

Answers to subquestions were either already given in the solutions to the 2021 Final or are not relevant for the 2023 course. The only exception is a) and e):

a) A sequence (f_n) is Cauchy in $C^0[0,1]$ if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|f_n - f_m|_{\max} < \varepsilon$. Using the definition of $|\cdot|_{\max}$ this can be reformulated as

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\ \forall\, n, m \geq N,\ |f_n(x) - f_m(x)| < \varepsilon,\ \forall\, x \in [0,1]\,.$$

b) There is a typo in the exercise as stated: $R(x) = f(x) - 1$ and not $R(x) = f(x) - x$. Saying that $f(0) = 1$ and $f'(0) = 0$ is the same as saying that $R(x) = \rho(x)x$ where ρ is continuous at $x = 0$ and $\rho(0) = 0$. This is equivalent to saying that for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|x| < \delta$, then $|R(x)| < \varepsilon|x|$.

Solution to Problem 2.

Rewrite the equation as $x = \frac{1}{2} + \frac{1}{2}\sin x$. Consider the function $f\colon [0,1] \to [0,1]$ such that $f(x) = \frac{1}{2} + \frac{1}{2}\sin x$ for all $x \in [0,1]$. The target of the function is indeed correct since $|f(x) - \frac{1}{2}| = |\frac{1}{2}\sin x| \leq \frac{1}{2}$. By the Banach contraction principle, the desired result follows by showing that f is a contraction. If $x, y \in [0,1]$ with $x < y$, then $|f(x) - f(y)| = |f'(c)||x - y|$ for some $c \in (x, y)$ as follows from the mean-value theorem since f is differentiable. On the other hand, $|f'(c)| = \frac{1}{2}|\cos(c)| \leq \frac{1}{2}$. Thus, $|f(x) - f(y)| \leq r|x - y|$ with $r = \frac{1}{2} \in [0,1)$ and hence f is a contraction.

Solution to Problem 3.

We use the integrability criterion from Problem 1b. For each $N \in \mathbb{N}$ consider the partition P_N which divides $[0,1]$ into N intervals of length $1/N$, i.e., $x_n - x_{n-1} = \frac{1}{N}$ for all $n = 1, \dots, N$. For all $x, y \in [x_{n-1}, x_n]$, we have $f(x) - f(y) \leq |x - y| \leq x_n - x_{n-1} = \frac{1}{N}$. Therefore, $M_n - m_n \leq \frac{1}{N}$ and

$$\overline{S} - \underline{S} = \sum_{n=1}^{N} (M_n - m_n)(x_n - x_{n-1}) \leq \sum_{n=1}^{N} \frac{1}{N}(x_n - x_{n-1}) = \frac{1}{N}(x_N - x_0) = \frac{1}{N}\,.$$

Thus, given ε, it is enough to choose N such that $\frac{1}{N} < \varepsilon$ to find P_N such that $\overline{S} - \underline{S} < \varepsilon$.

Solution to Problem 4.

There is a typo and should be $|x|^{\frac{5}{4}}$ instead of $x^{\frac{5}{4}}$. We have $R(x) = f(x) - 1 = \frac{|x|^{\frac{5}{4}}}{1+x^4}$. Therefore, $|R(x)| = \left|\frac{|x|^{\frac{1}{4}}}{1+x^4}\right| |x|$. Given $\varepsilon > 0$, let $\delta > 0$ to be found below and estimate for all $x \in \mathbb{R}$ with $|x| < \delta$:

$$\left|\frac{|x|^{\frac{1}{4}}}{1+x^4}\right| \leq \frac{|x|^{\frac{1}{4}}}{1+0} < \delta^{\frac{1}{4}}\,.$$

If δ is such that $\delta^{\frac{1}{4}} = \varepsilon$, then δ is a suitable response to ε. This means $\delta := \varepsilon^4$.

Solution to Problem 5.

We just prove that Φ is a contraction (the fact that there is a unique fixed points follows then from the Banach contraction principle for complete metric spaces which we didn't have time to discuss).
Let $f, g \in C[0,1]$ and let $x \in [0,1]$. Then,

$$\begin{aligned}\left|\Phi(f)(x) - \Phi(g)(x)\right| = \left|3 + \int_0^x \frac{f(t)}{2+t}\mathrm{d}t - \left(3 + \int_0^x \frac{g(t)}{2+t}\mathrm{d}t\right)\right| &= \left|\int_0^x \frac{f(t)-g(t)}{2+t}\mathrm{d}t\right| \\ &\leq \int_0^x \frac{|f(t)-g(t)|}{2+t}\mathrm{d}t \\ &\leq \int_0^x \frac{|f(t)-g(t)|}{2+0}\mathrm{d}t \\ &\leq \tfrac{1}{2}\int_0^x |f-g|_{\max}\mathrm{d}t \\ &= \tfrac{1}{2}x|f-g|_{\max} \\ &\leq \tfrac{1}{2}|f-g|_{\max}\,,\end{aligned}$$

where we used linearity and triangular inequality of the integral, monotonicity of the integral combined with $\frac{1}{1+t} \leq 1$ and $|f(t) - g(t)| \leq |f - g|_{\max}$ for all $t \in [0,1]$, the integral of a constant and the fact that $x \leq 1$. Since this is true for all x, then $|\Phi(f) - \Phi(g)|_{\max} = \max_{0\leq x\leq 1} |\Phi(f)(x) - \Phi(g)(x)| \leq \frac{1}{2}|f - g|_{\max}$.

Solution to Problem 6.

This problem is not relevant for the course of 2023.

Solution to Problem 7.

The hypothesis is that there are some ε, δ such that $\forall\, k \in \mathbb{N}$ and $\forall\, x, y \in [0,1]$ we have that $|x-y| < \delta$ implies $|f_k(x) - f_k(y)| < \varepsilon$.
By the Archimedean property, there is $M \in \mathbb{N}$ such that $\frac{1}{M} < \delta$. Consider the numbers $q_j := \frac{j}{M} \in \mathbb{Q} \cap [0,1]$ with $j = 0, \ldots, M$. For each fixed j, the sequence $(f_n(q_j))_{n\in\mathbb{N}}$ is Cauchy. Therefore, there is $N_j \in \mathbb{N}$ such that for all $m, n \geq N_j$ we have $|f_n(q_j) - f_m(q_j)| < \varepsilon$. Let $N := \max\{N_1, \ldots, N_M\} \in \mathbb{N}$. Let $n, m \geq N$. Take $t \in [0,1]$ arbitrary. Then, there exists $j \in \{1, \ldots, M\}$ such that $t \in [q_{j-1}, q_j]$. This implies that $|t - q_j| < \frac{1}{M} < \delta$. Applying the hypotheses with $x = t$, $y = q_j$ and $k = n, m$ we see that $|f_n(t) - f_n(q_j)| < \varepsilon$ and $|f_m(t) - f_m(q_j)| < \varepsilon$. Therefore,

$$|f_n(t) - f_m(t)| \leq |f_n(t) - f_n(q_j)| + |f_n(q_j) - f_m(q_j)| + |f_m(q_j) - f_m(t)| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon\,,$$

where we used triangular inequality twice and the fact that $|f_n(q_j) - f_m(q_j)| < \varepsilon$ since $n, m \geq N \geq N_j$.