Vrije Universiteit Amsterdam | **May 28 2021**
Mathematical Analysis, full and final exam | **First examiner: Joost Hulshof**
16.00-19.00 | **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers. You can and *will have to use earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed.

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7 problems worth $2+1+1+1+1+\frac{3}{2}+\frac{3}{2}=9$. Your grade will be $1+T$, T your total score, maximal $T=9$.

Problem 1. (2 points) Some basic theory needed for Problems 2,3,4,6,7.

a) ($\frac{1}{3}$ point) Give the ε, N-definition for the sequence x_n in $\mathbb{R}$ to be a Cauchy sequence.

b) ($\frac{1}{3}$ point) A function $f : [0,1] \to \mathbb{R}$ is called uniformly continuous on $[0,1]$ if

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{x,y\in[0,1]} : |x-y|<\delta \implies |f(x)-f(y)|<\varepsilon.$$

Give the logical negation of this ε, δ-statement.

c) ($\frac{1}{3}$ point) Let $f : [0,1] \to \mathbb{R}$ be a bounded function. For a partition P given by $N \in \mathbb{N}$ and points

$$0 = x_0 \le x_1 \le \cdots \le x_N = 1$$

we write

$$\sum_{n=1}^{N} m_n(x_n - x_{n-1}) = \underline{S} \le \bar{S} = \sum_{n=1}^{N} M_n(x_n - x_{n-1})$$

in which

$$m_n = \inf_{x\in[x_{n-1},x_n]} f(x), \quad M_n = \sup_{x\in[x_{n-1},x_n]} f(x).$$

Formulate an ε-statement that characterises the integrability of f on $[0,1]$ in terms of differences $\bar{S}-\underline{S}$.

d) ($\frac{1}{3}$ point) Formulate the Banach Fixed Point Theorem for maps $f : [0,1] \to [0,1]$.

e) ($\frac{1}{3}$ point) Let $f : \mathbb{R} \to \mathbb{R}$ and let $R(x) = f(x) - x$ for $x \in \mathbb{R}$.
Give the ε, δ-statement on the remainder term $R(x)$ for f to be differentiable in $x = 0$ with $f'(0) = 1$.

f) ($\frac{1}{3}$ point) Let (a, b) be an open non-empty interval in $\mathbb{R}$, and let $f \in C([a, b])$ be differentiable on (a, b).
Formulate the Mean Value Theorem for f on $[a, b]$.

Problem 2. (1 point) Prove that $x = \cos x$ has a unique solution in $[0,1]$. Hint: use 2 items of Problem 1. You may also use what you know about cos and sin from calculus.

Problem 3. (1 point) Let $f : [0,1] \to \mathbb{R}$ be nondecreasing. Prove that f is integrable on $[0,1]$.

Problem 4. (1 point) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x + \frac{x^{\frac{4}{3}}}{1+x^2}$$

for all $x \in \mathbb{R}$. Use the ε, δ-statement for the remainder term to show that f is differentiable in $x = 0$.

With everything correct so far your grade will be at least 6.

Recall $C([0,1])$, the space of all real valued continuous functions on $[0,1]$, complete with respect to the metric d defined by the maximum norm:

$$d(f,g) = \max_{0\leq x\leq 1} |f(x) - g(x)| = |f - g|_{max}, \quad f,g \in C([0,1]).$$

Problem 5. (1 point) Define $\Phi : C([0,1]) \to C([0,1])$ by

$$\Phi(f)(x) = 1 + \frac{1}{2}\int_0^x \frac{f(s)}{1+s}\,ds$$

for all $x \in [0,1]$ and all $f \in C([0,1])$. You don't have to prove that Φ is well defined. Prove that

$$|\Phi(f) - \Phi(g)|_{max} \leq \frac{1}{2}|f-g|_{max}$$

for all $f, g \in C([0,1])$.

Problem 6. Let $p(x)$ be a power series of the form

$$p(x) = 1 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots,$$

in which the coefficients a_{2n} indexed by $n \in \mathbb{N}$ are all positive.

a) ($\frac{1}{2}$ point) Find an expression for a_{2n}, $n \in \mathbb{N}$, if it is given that

$$p''(x) = p(x)$$

for every $x \in [0,1]$.

Write f_n for the function defined by

$$f_n(x) = 1 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots + a_{2n}x^{2n} = 1 + \sum_{k=1}^{n} a_{2k}x^{2k}$$

for all $x \in [0,1]$.

b. ($\frac{1}{2}$ point) Show that $f_n(1)$ is a convergent sequence in $\mathbb{R}$.
Hint: use that $f_n(1)$ is an increasing sequence and estimate a_n by a suitable power.
In case you don't have a) take $a_n = \frac{1}{(2n+1)!}$.

c. ($\frac{1}{2}$ point) Use the previous item to show that f_n is a convergent sequence in $C([0,1])$.
Hint: for $n > m$ write

$$f_n(x) - f_m(x) = \sum_{k=m+1}^{n} a_{2k}x^{2k}$$

and estimate to show that f_n is a Cauchy sequence with respect to the maximum norm.

Problem 7. In this exercise you will prove that every $f \in C([0,1])$ is uniformly continuous on $[0,1]$. To do so let $f : [0,1] \to \mathbb{R}$ be a function which is not uniformly continuous on $[0,1]$.

a) ($\frac{1}{2}$ point) Prove there exist $\varepsilon > 0$ and sequences x_n, y_n in $[0,1]$ with $|f(x_n) - f(y_n)| \geq \varepsilon$ and $x_n - y_n \to 0$.

b) ($\frac{1}{2}$ point) The sequence x_n has a convergent subsequence x_{n_k} with limit ξ in $[0,1]$.
Prove that $y_{n_k} \to \xi$ as $k \to \infty$ with an ε-argument.

c) ($\frac{1}{2}$ point) Use the definition of continuity with sequences to show that f is not continuous in ξ.

This completes the proof.