VU Amsterdam	Mathematical Analysis (XB_0009)
Faculty of Sciences	Solutions to Final Exam 2021
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Solution to Problem 1.

a) A sequence is Cauchy if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|x_n - x_m| < \varepsilon$.

b) The logical negation to uniform continuity is:

$$\exists \varepsilon_0 > 0\ \forall \delta > 0,\ \exists x, y \in [0,1],\ |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon_0 .$$

c) The function f is integrable if for all $\varepsilon > 0$ there is a partition P such that $\overline{S} - \underline{S} < \varepsilon$.

d) If a function $f\colon [0,1] \to [0,1]$ is a contraction, there exists a unique $\alpha \in [0,1]$ such that $f(\alpha) = \alpha$. Addendum: Given any sequence (x_n) defined by $x_0 \in [0,1]$ and $x_{n+1} = f(x_n)$, we have $|\alpha - x_n| \leq \frac{r^n}{1-r}|x_1 - x_0|$ for all n, where r is the contraction factor of f.

e) (I think here the assumption $f(0) = 0$ is missing). Let $R(x) = f(x) - x$. Saying that $f(0) = 0$ and $f'(0) = 1$ is the same as saying that $R(x) = \rho(x)x$ where ρ is continuous at $x = 0$ and $\rho(0) = 0$. This is equivalent to saying that for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|x| < \delta$, then $|R(x)| < \varepsilon|x|$.

f) If f is continuous on $[a,b]$ and differentiable on (a,b), then there is $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Solution to Problem 2.

Consider the function $f\colon [0,1] \to [0,1]$ such that $f(x) = \cos x$ for all $x \in [0,1]$. The target of the function is indeed correct since $|\cos x| \leq 1$ and $\cos(x) \geq 0$ for all $x \in [0, \frac{\pi}{2}] \supset [0,1]$. By the Banach contraction principle, the desired result follows by showing that f is a contraction. If $x, y \in [0,1]$ with $x < y$, then $|f(x) - f(y)| = |f'(c)||x - y|$ for some $c \in (x,y)$ as follows from the mean-value theorem since f is differentiable. On the other hand, $|f'(c)| = |-\sin(c)| = |\sin(c)| = \sin c \leq \sin 1$ where we have used that sin is non-negative and increasing in the interval $[0,1]$. Thus, $|f(x) - f(y)| \leq r|x - y|$ with $r = \sin(1)$ since $1 \in [0, \frac{\pi}{2})$ we see that $\sin(1) < 1$ and hence f is a contraction.

Solution to Problem 3.

This was proved for Assignment 5 in year 2023.

Solution to Problem 4.

We have $R(x) = f(x) - x = \frac{x^{\frac{4}{3}}}{1+x^2}$. Therefore, $|R(x)| = \left|\frac{x^{\frac{1}{3}}}{1+x^2}\right| |x|$. Given $\varepsilon > 0$, let $\delta > 0$ to be found below and estimate for all $x \in \mathbb{R}$ with $|x| < \delta$:

$$\left|\frac{x^{\frac{1}{3}}}{1+x^2}\right| \leq \frac{|x|^{\frac{1}{3}}}{1+0} < \delta^{\frac{1}{3}} .$$

If δ is such that $\delta^{\frac{1}{3}} = \varepsilon$, then δ is a suitable response to ε. This means $\delta := \varepsilon^3$.

Solution to Problem 5.

Let $f, g \in C[0,1]$ and let $x \in [0,1]$. Then,

$$\begin{aligned}
\big|\Phi(f)(x) - \Phi(g)(x)\big| = \left|1 + \tfrac{1}{2}\int_0^x \frac{f(t)}{1+t}\mathrm{d}t - \left(1 + \tfrac{1}{2}\int_0^x \frac{g(t)}{1+t}\mathrm{d}t\right)\right| &= \tfrac{1}{2}\left|\int_0^x \frac{f(t)-g(t)}{1+t}\mathrm{d}t\right| \\
&\leq \tfrac{1}{2}\int_0^x \frac{|f(t)-g(t)|}{1+t}\mathrm{d}t \\
&\leq \tfrac{1}{2}\int_0^x |f(t)-g(t)|\mathrm{d}t \\
&\leq \tfrac{1}{2}\int_0^x |f-g|_{\max}\mathrm{d}t \\
&= \tfrac{1}{2}x|f-g|_{\max} \\
&\leq \tfrac{1}{2}|f-g|_{\max}\,,
\end{aligned}$$

where we used linearity and triangular inequality of the integral, monotonicity of the integral combined with $\frac{1}{1+t} \leq 1$ and $|f(t)-g(t)| \leq |f-g|_{\max}$ for all $t \in [0,1]$, the integral of a constant and the fact that $x \leq 1$. Since this is true for all x, then $|\Phi(f)-\Phi(g)|_{\max} = \max_{0\leq x\leq 1}|\Phi(f)(x)-\Phi(g)(x)| \leq \frac{1}{2}|f-g|_{\max}$.

Solution to Problem 6.

This problem is not relevant for the course of 2023.

Solution to Problem 7.

This problem was proved as a Theorem in Lecture 8 of 2023.