Vrije Universiteit Amsterdam | **March 26 2021**
Mathematical Analysis, midterm exam | **First examiner: Joost Hulshof**
12.15-14.15 | **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers. You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed.

Every item below is $\frac{1}{2}$ point, except 2a, 3c, 5a, which are 1 point.
Your grade will be $1 + T$, T your total score, maximal $T = 9$.

Problem 1. Some basic theory needed for the next exercises.

a) ($\frac{1}{2}$ point) Formulate the Archimedean Principle.

b) ($\frac{1}{2}$ point) Give the definition of a convergent sequence.

c) ($\frac{1}{2}$ point) Formulate the Banach Fixed Point Theorem for real valued functions of a real variable.

Problem 2. Consider the sequence x_n indexed by $n \in \mathbb{N}$ and defined by

$$x_n = \frac{n}{n^2 + 1}.$$

a) (1 point) Prove that 0 is the largest lower bound for the sequence x_n.

b) ($\frac{1}{2}$ point) Prove that x_n is convergent.

Problem 3. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f(x) = x + \frac{1}{x}$$

You can use the continuity of f only after the first question.

Let $\xi \in \mathbb{R}^+$ and let $x_n \in \mathbb{R}^+$ be a sequence indexed by $n \in \mathbb{N}$ with $x_n \to \xi$ as $n \to \infty$.

a) ($\frac{1}{2}$ point) Use the ε-definition to prove that $f(x_n) \to f(\xi)$ as $n \to \infty$.

Let $A = \{f(x) : x \in \mathbb{R}^+\}$ be the image of $\mathbb{R}^+$ under f.
Then A is non-empty and bounded from below by 0.
Therefore there exists a largest lower bound $m \geq 0$ for A.

b) ($\frac{1}{2}$ point) Prove that there exists a sequence $y_n \in A$ with $y_n \to m$ as $n \to \infty$.

By definition of A we have for every n that $y_n = f(x_n)$ for some $x_n \in \mathbb{R}^+$.
So with the sequence y_n we also have a sequence x_n.
Like any sequence of real numbers, x_n has a monotone subsequence x_{n_k}, indexed by $k \in \mathbb{N}$.

c) (1 point) Show that $x_{n_k} \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^+$ as $k \to \infty$.
Hint: distinguish between x_{n_k} non-decreasing and x_{n_k} non-increasing.

Define x_n by $x_0 = 1$ and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. Then x_n is a strictly increasing sequence.

d) ($\frac{1}{2}$ point) Is x_n a convergent sequence?
Hint: use the continuity of f.

Problem 4. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f(x) = \frac{x}{3} + \frac{1}{x}$$

a) ($\frac{1}{2}$ point) For which $a > 0$ is f contractive on $[a, \infty)$?

For such a and $x \geq a > 0$ we have

$$f(x) \geq \frac{a}{3} + \frac{1}{x},$$

so $f(x) \geq a$ provided

$$\frac{a}{3} + \frac{1}{x} \geq a.$$

Let b be the largest x for which this latter inequality holds.

b) ($\frac{1}{2}$ point) For which a is f a map from $[a, b]$ to itself?
Hint: Express b in a and estimate both terms in $f(x)$ from above.

c) ($\frac{1}{2}$ point) Prove that $f(x) = x$ has a positive solution x.

Problem 5. Let $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f_n(x) = \frac{x}{n} + \frac{1}{x}$$

a) (1 point) Show that the sequence f_n is uniformly convergent on $(0, 1]$.

b) ($\frac{1}{2}$ point) Is the sequence f_n uniformly convergent on $\mathbb{R}^+$? Explain!

c) ($\frac{1}{2}$ point) Let x_n be the solution of $f_n(x) = x$. Is the sequence x_n convergent?
If yes, use your calculus skills to compute the limit.