Vrije Universiteit Amsterdam | **March 26 2021**
Mathematical Analysis, midterm exam | **First examiner: Joost Hulshof**
12.15-14.15 | **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers. You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed.

Every item below is $\frac{1}{2}$ point, except 2a, 3c, 5a, which are 1 point.
Your grade will be $1 + T$, T your total score, maximal $T = 9$.

Problem 1. Some basic theory needed for the next exercises.

a) ($\frac{1}{2}$ point) Formulate the Archimedean Principle.

b) ($\frac{1}{2}$ point) Give the definition of a convergent sequence.

c) ($\frac{1}{2}$ point) Formulate the Banach Fixed Point Theorem for real valued functions of a real variable.

Problem 2. Consider the sequence x_n indexed by $n \in \mathbb{N}$ and defined by

$$x_n = \frac{n}{n^2 + 1}.$$

a) (1 point) Prove that 0 is the largest lower bound for the sequence x_n.

b) ($\frac{1}{2}$ point) Prove that x_n is convergent.

Answers:

a) Clear 0 is lower bound for the sequence. Let $m \geq 0$ be the largest lower bound and suppose that $m > 0$. To establish a contradiction we look for an $n \in \mathbb{N}$ with $x_n < m$. So we need

$$\frac{n}{n^2 + 1} < m$$

Use

$$\frac{n}{n^2 + 1} < \frac{n}{n^2} = \frac{1}{n}$$

and invoke the Archimedean Principle for the existence of $n \in \mathbb{N}$ with

$$\frac{1}{n} < m.$$

Then

$$x_n = \frac{n}{n^2 + 1} < \frac{1}{n} < m,$$

contradicting the assumption that m is a lower bound.

b) We use the same estimate for x_n. Let $\varepsilon > 0$ and invoke the Archimedean Principle for the existence of $N \in \mathbb{N}$ with

$$\frac{1}{N} < \varepsilon.$$

For all $n \geq N$ it then holds that

$$|x_n - 0| = \frac{n}{n^2 + 1} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

which completes the proof.

Alternatively, show that x_n is decreasing and use the first part.

Problem 3. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f(x) = x + \frac{1}{x}$$

You can use the continuity of f only after the first question.

Let $\xi \in \mathbb{R}^+$ and let $x_n \in \mathbb{R}^+$ be a sequence indexed by $n \in \mathbb{N}$ with $x_n \to \xi$ as $n \to \infty$.

a) ($\frac{1}{2}$ point) Use the ε-definition to prove that $f(x_n) \to f(\xi)$ as $n \to \infty$.

Let $A = \{f(x) : x \in \mathbb{R}^+\}$ be the image of $\mathbb{R}^+$ under f.
Then A is non-empty and bounded from below by 0.
Therefore there exists a largest lower bound $m \geq 0$ for A.

b) ($\frac{1}{2}$ point) Prove that there exists a sequence $y_n \in A$ with $y_n \to m$ as $n \to \infty$.

By definition of A we have for every n that $y_n = f(x_n)$ for some $x_n \in \mathbb{R}^+$.
So with the sequence y_n we also have a sequence x_n.
Like any sequence of real numbers, x_n has a monotone subsequence x_{n_k}, indexed by $k \in \mathbb{N}$.

c) (1 point) Show that $x_{n_k} \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^+$ as $k \to \infty$.
Hint: distinguish between x_{n_k} non-decreasing and x_{n_k} non-increasing.

Define x_n by $x_0 = 1$ and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. Then x_n is a strictly increasing sequence.

d) ($\frac{1}{2}$ point) Is x_n a convergent sequence?
Hint: use the continuity of f.

Answers:

a) *NB You are actually being asked to prove that f is continuous in ξ using the definition of continuity with limits of sequences.*
Let $\varepsilon > 0$. Then you need an estimate of the form

$$|f(x_n) - f(\xi)| = |x_n + \frac{1}{x_n} - \xi - \frac{1}{\xi}| < \varepsilon$$

for $n \geq N$, N to be found using the ε-characterisation of $x_n \to \xi$.
Since

$$|x + \frac{1}{x} - \xi - \frac{1}{\xi}| = |x - \xi + \frac{1}{x} - \frac{1}{\xi}| \leq |x - \xi| + |\frac{1}{x} - \frac{1}{\xi}| \leq |x - \xi| + \frac{|x - \xi|}{x\xi} = (1 + \frac{1}{x\xi})\,|x - \xi|$$

for $x > 0$ we have

$$|f(x_n) - f(\xi)| < (1 + \frac{1}{x_n\xi})\,|x_n - \xi| < (1 + \frac{1}{x_n\xi})\,\varepsilon$$

for all $n \geq N$, N provided by the ε-characterisation of $x_n \to \xi$. This N depends on $\varepsilon > 0$.
We next look for a bound on the prefactor

$$1 + \frac{1}{x_n\xi}$$

and use the ε-characterisation of $x_n \to \xi$ with $\varepsilon = \frac{\xi}{2}$ to obtain N_1 such that $|x_n - \xi| < \frac{\xi}{2}$ for all $n \geq N_1$.
In particular $x_n > \frac{\xi}{2}$ and

$$1 + \frac{1}{x_n\xi} < 1 + \frac{2}{\xi^2}.$$

It follows for $n \geq \max(N, N_1)$ that

$$|f(x_n) - f(\xi)| < (1 + \frac{2}{\xi^2})\,\varepsilon.$$

This proves the statement that

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N : \quad |f(x_n) - f(\xi)| < (1 + \frac{2}{\xi^2})\,\varepsilon.$$

Now redo the argument starting with

$$\tilde{\varepsilon} = \frac{\varepsilon}{1 + \frac{2}{\xi^2}}$$

to conclude.

b) Let $n \in \mathbb{N}$. Since m is the largest lower bound for A the number $m + \frac{1}{n}$ is not a lower bound and thus there exists $y_n \in A$ such that $y_n < m + \frac{1}{n}$. But then $m \leq y_n < m + \frac{1}{n}$. Thus $|y_n - m| < \frac{1}{n}$ and as in 2b this implies that $y_n \to m$ as $n \to \infty$.

c) If x_{n_k} is non-decreasing then it may be bounded or not bounded from above. If it is then it converges to it lowest upper bound as $k \to \infty$.

It it is not bounded then $y_{n_k} = f(x_{n_k}) > x_{n_k}$ is also unbounded, contradicting $y_{n_k} \to m$ as $k \to \infty$.

If x_{n_k} is non-increasing then it has its largest lower bound as limit. Call that limit $\bar{x}$. It remains to show that $\bar{x} > 0$. If not then it must be that $\bar{x} = 0$ and then

$$y_{n_k} = f(x_{n_k}) > \frac{1}{x_{n_k}}$$

NB Not asked is: call that limit $\bar{x}$. By a) it follows that $y_{n_k} = f(x_{n_k}) \to f(\bar{x})$ as $k \to \infty$. But by b) $y_{n_k} \to f(\bar{x})$. It follows that $f(\bar{x}) = m$.

d) You don't have to check that x_n is increasing. Can it be bounded? If so it has a limit $\bar{x} > 0$ and it follows that $x_n \to \bar{x}$ and thereby $f(x_n) \to f(\bar{x})$. But $f(x_n) = x_{n+1} \to \bar{x}$ so $\bar{x} = f(\bar{x})$. For this f this would say that

$$\bar{x} = \bar{x} + \frac{1}{\bar{x}}$$

which is a contradiction.

Problem 4. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f(x) = \frac{x}{3} + \frac{1}{x}$$

a) ($\frac{1}{2}$ point) For which $a > 0$ is f contractive on $[a, \infty)$?

For such a and $x \geq a > 0$ we have

$$f(x) \geq \frac{a}{3} + \frac{1}{x},$$

so $f(x) \geq a$ provided

$$\frac{a}{3} + \frac{1}{x} \geq a.$$

Let b be the largest x for which this latter inequality holds.

b) ($\frac{1}{2}$ point) For which a is f a map from $[a, b]$ to itself?

Hint: Express b in a and estimate both terms in $f(x)$ from above.

c) ($\frac{1}{2}$ point) Prove that $f(x) = x$ has a positive solution x.

Anwers:

a) We have

$$f(x) - f(y) = (\frac{1}{3} - \frac{1}{xy})(x - y)$$

and

$$(\frac{1}{3} - \frac{1}{a^2}) \leq (\frac{1}{3} - \frac{1}{xy}) < \frac{1}{3}$$

Thus f is contractive if

$$\frac{1}{3} - \frac{1}{a^2} \geq -1,$$

which is equivalent to $a^2 > \frac{3}{4}$.

b) Compute $b = \frac{3}{2a}$.

For all $x \geq a$ we then have

$$f(x) \geq \frac{a}{3} + \frac{1}{x} \geq \frac{a}{3} + \frac{1}{b} = a,$$

and also

$$f(x) = \frac{x}{3} + \frac{1}{x} \leq \frac{b}{3} + \frac{1}{a} = \frac{1}{2a} + \frac{1}{a} = b.$$

So $a \leq f(x) \leq b$ for all $x \in [a, b]$.

c) Choose a as in a) such that $b > a$. The latter requires $\frac{3}{2a} > a$, i.e. $a^2 < \frac{3}{2}$. With $a^2 > \frac{3}{4}$ in a) it follows that f is a contraction from $[a, \frac{3}{2a}]$ to itself if $\frac{3}{4} < a^2 < \frac{3}{2}$. Invoke 1c.

Problem 5. Let $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f_n(x) = \frac{x}{n} + \frac{1}{x}$$

a) (1 point) Show that the sequence f_n is uniformly convergent on $(0, 1]$.

b) ($\frac{1}{2}$ point) Is the sequence f_n uniformly convergent on $\mathbb{R}^+$? Explain!

c) ($\frac{1}{2}$ point) Let x_n be the solution of $f_n(x) = x$. Is the sequence x_n convergent? If yes, use your calculus skills to compute the limit.

Answers

a) The pointwise limit is $f(x) = x$. For uniform convergence estimate

$$|f_n(x) - f(x)| = |\frac{x}{n}| \le \frac{1}{n}.$$

So given $\varepsilon > 0$ choose $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$, then for all $n \ge N$

$$|f_n(x) - f(x)| = |\frac{x}{n}| \le \frac{1}{n} \le \frac{1}{N} < \varepsilon.$$

b) There's now way to get

$$|f_n(x) - f(x)| = |\frac{x}{n}| < \varepsilon$$

for all $x > 0$ simultaneously. Not for any $\varepsilon > 0$ and $n \in \mathbb{N}$. Just take $x = n\varepsilon$ to see why. So no uniform convergence on $\mathbb{R}^+$.

c) Just solve the equation to find

$$x_n = \sqrt{\frac{n}{n-1}} \to 1$$

as $n \to \infty$.