Vrije Universiteit Amsterdam — **July 2020**

Mathematical Analysis, open course notes take home resit — **First examiner: Joost Hulshof**

3 hours — **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers (refer to theorems used). You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed, except the course notes, use them! Your grade will be $1+\frac{T}{4}$, T your total score.

Problem 1. For $\mathbf{a+b+c=3+3+3=9}$ **points** you have to give your answers in this exercise with epsilon arguments. Let $F:\mathbb{R}\to\mathbb{R}$ be defined by

$$F(x)=\frac{x^3}{1-x^3}\quad\text{for all}\quad x\neq 1\quad\text{and}\quad F(1)=-1.$$

a) Prove that F is discontinuous in $x=1$.

b) For every $n\in\mathbb{N}$ we define $f_n:(0,1)\to\mathbb{R}$ by

$$f_n(x)=F(\frac{x}{n})\quad\text{for all}\quad x\in(0,1).$$

Prove that $f_n(x)$ is a convergent sequence for every $x\in(0,1)$.

c) Prove that the convergence is not uniform on $(-1,1)$.

Problem 2. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=2x+\frac{1}{3}x^3.$$

This f can next earn you

$$\mathbf{a+b+c+d=2+1+2+4=9\ points}$$

a) Use epsilon-delta arguments for the appropriate remainder term to show f is differentiable in $x=0$.

b) Now consider for $y\in\mathbb{R}$ fixed the equation $f(x)=y$ and the scheme $x_n=x_{n-1}+f'(0)^{-1}(y-f(x_{n-1}))$ to solve $f(x)=y$. Verify that

$$x_n=\frac{1}{2}y-\frac{1}{6}x_{n-1}^3. \tag{1}$$

c) Starting from $x_0=0$ the scheme (1) defines a sequence x_n. Suppose that for some $n\in\mathbb{N}$ it holds that

$$|x_{n-1}|\leq 1\quad\text{and}\quad |x_n|\leq 1\,.$$

Use (1) to show that

$$|x_{n+1}-x_n|\leq\frac{1}{2}\,|x_n-x_{n-1}|\,. \tag{2}$$

d) The sequence x_n depends on y, and thereby defines a sequence of functions g_n by setting $g_n(y)=x_n$. Show there exists $r>0$ such that g_n is a uniform Cauchy sequence in $C([-r,r])$.

Problem 3. For **a + b + c + d = 2 + 2 + 2 + 2 = 8 points** consider solutions of

$$f''(x) + \frac{1}{x}\, f'(x) + f(x) = 0,$$

a differential equation posed for $x > 0$ first here.

a) Suppose that
$$f(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \cdots$$
is a power series solution defined for all x in some interval $(0, r)$. Show that
$$a_{2n} = -\frac{a_{2n-2}}{(2n)^2}$$
for all $\in \mathbb{N}$.

b) Fix $x > 0$. Use a) *and an estimate for n sufficiently large* to establish convergence of the series.

c) Let f be any solution of the differential equation defined on an open interval contained in $\mathbb{R}_+$ and let
$$E(x) = f'(x)^2 + f(x)^2.$$
Show that $E'(x) \leq 0$ on that interval. Hint: use the differential equation and not the power series expansion of its solution when you evaluate $E'(x)$.

d) By b) the power series in a) is a solution that satisfies $f(x) \to 1$ and $f'(x) \to 0$ as $x \to 0$.
Show there are no other solutions on $\mathbb{R}_+$ with this property. Hint: if $g(x)$ is another such solution then $v(x) = f(x) - g(x)$ is also a solution of the differential equation. Apply c) to v.

Problem 4. For more than **a + b + c + d + e = 2 + 2 + 2 + 2 + 2 = 10 points** we consider the differential equation $f'(x) = 1 + f(x)^2$ with initial value $f(0) = 0$.

a) Let $r > 0$ and suppose that $f \in C([0, r])$ is a solution. Integrate the differential equation to show that
$$f(x) = \int_0^x (1 + f(s)^2)\, ds \quad \text{for all} \quad x \in [0, r]. \tag{3}$$

b) Denote the right hand side of (3) by $(\Phi(f))(x)$.
Explain why this defines a map $\Phi : C([0, r]) \to C([0, r])$.

c) Voor $f \in C([0, r])$ we write
$$|f|_r = \max_{x \in [0,r]} |f(x)|$$
for the maximum norm of f. Show that
$$|\Phi(f) - \Phi(g)|_r \leq r\,(|f|_r + |g|_r)\,|f - g|_r$$
for every $f, g \in C([0, r])$.

d) Let $r, R > 0$ and
$$A = A_{rR} = \{f \in C([0, r]) : |f|_r \leq R\}.$$
Show that
$$|\Phi(f)|_r \leq r\,(1 + R^2)$$
for every $f \in A$.

e) Show there exist $r > 0$ and $R > 0$ such that Φ is a contraction on A.

f) Bonus (2 points): describe the set of all $r > 0$ and $R > 0$ for which Φ is a contraction on A.