Vrije Universiteit Amsterdam — **July 2020**
Mathematical Analysis, open course notes take home resit — **First examiner: Joost Hulshof**
3 hours — **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers (refer to theorems used). You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed, except the course notes, use them! Your grade will be $1 + \frac{T}{4}$, T your total score.

Problem 1. For **a + b + c = 3 + 3 + 3 = 9 points** you have to give your answers in this exercise with epsilon arguments. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \frac{x^3}{1-x^3} \quad \text{for all} \quad x \neq 1 \quad \text{and} \quad F(1) = -1.$$

a) Prove that F is discontinuous in $x = 1$.

Answer. I reason as in the previous exam. We have

$$F(x) - F(1) = \frac{x^3}{1-x^3} + 1 = \frac{1}{1-x^3}$$

and see that for x close to 1 the denominator is small. Let's try to make it smaller than $\frac{1}{2}$ to obtain that for $\varepsilon = 2$ no $\delta > 0$ can be found with

$$(\Delta) \qquad |x-1| < \delta \implies |F(x) - F(1)| < 2.$$

So the x that you choose will depend on δ. We note that

$$|F(x) - F(1)| \geq 2 \iff |1-x^3| \leq \frac{1}{2} \iff \underbrace{|1-x^3| \leq \frac{1}{2}} \iff -\frac{1}{2} \leq x^3 - 1 \leq \frac{1}{2} \iff \frac{1}{2} \leq x^3 \leq \frac{3}{2},$$

and this certainly holds if for instance $\frac{9}{10} \leq x \leq \frac{11}{10}$ since then

$$\frac{1}{2} = \frac{500}{1000} < \frac{729}{1000} \leq x^3 \leq \frac{1331}{1000} < \frac{1500}{1000} = \frac{3}{2}.$$

Summing up we have for all x with $|x-1| < \frac{1}{10}$ that $|F(x) - F(1)| \geq 2$. This makes it impossible to have δ as in (Δ) above for $\varepsilon = 2$.

Etienne reasoned from the epsilon-delta statement and factorised to get

$$\varepsilon > |\frac{1}{1-x^3}| = \frac{1}{|x-1|\,|x^2+x+1|} > \frac{1}{\delta|x^2+x+1|}$$

and concluded this can not be. Correct, but needs more reasoning.

Other people took $\varepsilon = 1$ and reasoned from

$$|\frac{1}{1-x^3}| < 1 \iff |1-x^3| = |x^3-1| > 1.$$

The second one rewrites as

$$x^3 - 1 > 1 \quad \text{or} \quad x^3 - 1 < -1,$$

so either $x^3 > 2$ or $x^3 < 0$ must hold. The inequality $|F(x) - F(1)| < 1$ is thus equivalent with $x \notin [0, 2^{\frac{1}{3}}]$. That is: $|F(x) - F(1)| > 1$ for every $x \in [0, 2^{\frac{1}{3}}]$. But this interval has nonempty intersection with every interval $(1-\delta, 1+\delta)$. So for $\varepsilon = 1$ no $\delta > 0$ exists for which $|x-1| < \delta \implies |F(x) - F(1)| < 1$.

Per reasoned correctly from

$$|x-1| < 1 \implies |x^3 - 1| < 8,$$

which follows from

$$x^3 - 1 = (x^2+x+1)(x-1)$$

the factor $x^2 + x + 1$ being bounded in absolute value by 7 for $x \in (0, 2)$.

b) For every $n \in \mathbb{N}$ we define $f_n : (0,1) \to \mathbb{R}$ by

$$f_n(x) = F(\frac{x}{n}) \quad \text{for all} \quad x \in (0,1).$$

Prove that $f_n(x)$ is a convergent sequence for every $x \in (0,1)$.

Answer. *Fix $x \in (0,1)$. Since $x/n \to 0$ as $n \to \infty$ and F is continuous in 0 its is clear that the limit will be $F(0) = 0$. But you have to prove this with an ε-argument that ends with $|F(\frac{x}{n})| < \varepsilon$. Since*

$$0 < F(\frac{x}{n}) = \frac{(\frac{x}{n})^3}{1-(\frac{x}{n})^3} = \frac{x^3}{n^3 - x^3} < \frac{1}{n^3-1} < \varepsilon$$

for all $n > 1$ and the latter inequality is equivalent to

$$n^3 > 1 + \frac{1}{\varepsilon}$$

we can now give the proof: let $\varepsilon > 0$. Choose $N > 1 + \frac{1}{\varepsilon}$. Then for all $n \geq N$ it holds that for all $n > 1$ and the latter inequality is equivalent to

$$n^3 \geq N^3 \geq N > 1 + \frac{1}{\varepsilon}$$

and thereby

$$|F(\frac{x}{n})| < \varepsilon.$$

This completes the proof.

c) Prove that the convergence is uniform on $(-1,1)$.

This should have asked for uniform convergence. My mistake. Occasional bonus points.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2x + \frac{1}{3}x^3.$$

This f can next earn you

a + b + c + d = 2 + 1 + 2 + 4 = 9 points

a) Use epsilon-delta arguments for the appropriate remainder term to show f is differentiable in $x = 0$.

Answer. *With linear approximation $2x$ the remainder term is $R(x) = \frac{1}{3}x^3$. We have for $x \neq 0$ that*

$$|R(x)| < \varepsilon|x| \iff |\frac{1}{3}x^3| < \varepsilon|x| \iff |x^2| < 3\varepsilon,$$

so take $\delta = \sqrt{3\varepsilon}$ to conclude that $0 < |x| < \sqrt{3\varepsilon}$ implies $0 < |R(x)| < \varepsilon|x|$.

b) Now consider for $y \in \mathbb{R}$ fixed the equation $f(x) = y$ and the scheme $x_n = x_{n-1} + f'(0)^{-1}(y - f(x_{n-1}))$ to solve $f(x) = y$. Verify that

$$x_n = \frac{1}{2}y - \frac{1}{6}x_{n-1}^3. \tag{1}$$

Answer. *Use $f'(0) = 2$ and $f(x_{n-1}) = 2x_{n-1} + \frac{1}{3}x_{n-1}^3$. The linear terms in x_{n-1} drop out.*

c) Starting from $x_0 = 0$ the scheme (1) defines a sequence x_n. Suppose that for some $n \in \mathbb{N}$ it holds that

$$|x_{n-1}| \leq 1 \quad \text{and} \quad |x_n| \leq 1\,.$$

Use (1) to show that

$$|x_{n+1} - x_n| \leq \frac{1}{2}\,|x_n - x_{n-1}|\,. \tag{2}$$

Answer. *We have*

$$x_{n+1} - x_n = \frac{1}{6}x_{n-1}^3 - \frac{1}{6}x_n^3 = \frac{1}{6}\left(x_n^2 + x_n x_{n-1} + x_{n-1}^2\right)(x_{n-1} - x_n),$$

so

$$|x_{n+1}-x_n| \le \frac{1}{6}|x_n^2 + x_n x_{n-1} + x_{n-1}^2|\,|x_{n-1}-x_n|$$

$$\le \frac{1}{6}(\underbrace{|x_n|^2 + |x_n|\,|x_{n-1}| + |x_{n-1}|^2}_{\le 1+1+1})\,|x_{n-1}-x_n| \le \frac{1}{2}|x_{n-1}-x_n|.$$

d) The sequence x_n depends on y, and thereby defines a sequence of functions g_n by setting $g_n(y) = x_n$. Show there exists $r > 0$ such that g_n is a uniform Cauchy sequence in $C([-r, r])$.

Answer. I'll keep this short, see also the previous exam. Here we have

$$|x_n| = |s_1 + s_2 + \cdots + s_n| \le |s_1| + |s_2| + \cdots + |s_n| \le (1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}})|s_1| \le 2|s_1|$$

as long as all previous x_k *have* $|x_k| \le 1$. *So we have to make sure that* $|s_1| \le \frac{1}{2}$. *Since* $s_1 = x_1 = \frac{y}{2}$, *we see that with* $|y| \le 1$ *we're fine and obtain*

$$|s_n| \le \frac{|s_1|}{2^{n-1}} = \frac{|y|}{2\,2^{n-1}} \le \frac{1}{2^n}.$$

And then $m > n \ge N$ *gives*

$$|x_m - x_n| = |s_m + \cdots + s_n| \le |s_m| + \cdots + |s_{n+1}| < 2\,\frac{1}{2^{n+1}} = \frac{1}{2^n} \le \frac{1}{2^N},$$

so given $\varepsilon > 0$ *choose* $N \in \mathbb{N}$ *with* $2^N > \frac{1}{\varepsilon}$.

Problem 3. For **a + b + c + d = 2 + 2 + 2 + 2 = 8 points** consider solutions of

$$f''(x) + \frac{1}{x}\,f'(x) + f(x) = 0,$$

a differential equation posed for $x > 0$ first here.

a) Suppose that

$$f(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \cdots$$

is a power series solution defined for all x in some interval $(0, r)$. Show that

$$a_{2n} = -\frac{a_{2n-2}}{(2n)^2}$$

for all $\in \mathbb{N}$.

Answer. From the expression for f *we find*

$$f'(x) = 2a_2x + 4a_4x^3 + 6a_6x^5 + 8a_8x^7 + \cdots,$$

$$\frac{1}{x}\,f'(x) = 2a_2 + 4a_4x^2 + 6a_6x^3 + 8a_8x^6 + \cdots,$$

$$f''(x) = 2a_2 + 3\,4a_4x^2 + 5\,6a_6x^4 + 7\,8a_8x^6 + \cdots,$$

so sorting it out we find

$$f''(x)+\frac{1}{x}\,f'(x)+f(x) = a_0+2a_2+2a_2+(a_2+4a_4+3\,4a_4)x^2+(a_4+6a_6+5\,6a_6)x^4+(a_6+8a_8+7\,8a_8)x^6+\cdots,$$

which we put equal to zero by setting

$$a_0 + 2a_2 + 2a_2 = 0, \quad a_2 + 4a_4 + 3\,4a_4, \quad a_4 + 6a_6 + 5\,6a_6, \quad a_6 + 8a_8 + 7\,8a_8 = 0, \ldots,$$

whence

$$a_2 = -\frac{a_0}{4}, \quad a_4 = -\frac{a_2}{4^2}, \qquad a_6 = -\frac{a_4}{6^2}, \qquad a_8 = -\frac{a_6}{8^2}, \ldots,$$

and we recognise that every next a_{2n} *is minus the previous one divided by the square of* $2n$, *also for* $2n = 2$.

b) Fix $x > 0$. Use a) *and an estimate for n sufficiently large* to establish convergence of the series.

Answer. With $a_0 = 1$ we see that we get

$$a_{2n} = -\frac{1}{2^n (n!)^2},$$

and that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^2 x^{2n}}{2^n (n!)^2},$$

but that's not needed for the answer for which you should know to go for the simple geometric estimate

$$|a_{2n} x^{2n}| \leq \frac{1}{2} |a_{2n-2} x^{2n-2}|,$$

equivalent to

$$\frac{x^2}{(2n)^2} \leq \frac{1}{2},$$

as every next term $a_{2n}x^{2n}$ in the power series gets an additional x^2 in the numerator and $-(2n)^2$ in the denominator. So given x you have the desired estimate provided

$$n^2 \geq \frac{x^2}{2},$$

which certainly holds if $n \geq |x|$. Thus the sum of the terms from $n+1$ to any $m > n$ are bounded by the n^{th} term, which in turn goes to zero as $n \to \infty$ for x fixed, by the same reasoning.

c) Let f be any solution of the differential equation defined on an open interval contained in $\mathbb{R}_+$ and let

$$E(x) = f'(x)^2 + f(x)^2.$$

Show that $E'(x) \leq 0$ on that interval. Hint: use the differential equation and not the power series expansion of its solution when you evaluate $E'(x)$.

Answer. We have

$$E'(x) = 2f''(x)f'(x) + 2f'(x)f(x) = -2(f(x) + \frac{f'(x)}{x})f'(x) + 2f'(x)f(x) = -\frac{f'(x)^2}{x}$$

d) By b) the power series in a) is a solution that satisfies $f(x) \to 1$ and $f'(x) \to 0$ as $x \to 0$.

Show there are no other solutions on $\mathbb{R}_+$ with this property. Hint: if $g(x)$ is another such solution then $v(x) = f(x) - g(x)$ is also a solution of the differential equation. Apply c) to v.

Problem 4. For more than **a + b + c + d + e = 2 + 2 + 2 + 2 + 2 = 10 points** we consider the differential equation $f'(x) = 1 + f(x)^2$ with initial value $f(0) = 0$.

a) Let $r > 0$ and suppose that $f \in C([0,r])$ is a solution. Integrate the differential equation to show that

$$f(x) = \int_0^x (1 + f(s)^2)\, ds \quad \text{for all} \quad x \in [0,r]. \tag{3}$$

Answer. Use s as integration variable and write

$$f(x) = f(0) + \int_0^x f'(s)\, ds = \int_0^x (1 + f(s)^2)\, ds\, ds.$$

NB. You cannot evaluate the integral because you don't know f.

b) Denote the right hand side of (3) by $(\Phi(f))(x)$.

Explain why this defines a map $\Phi : C([0,r]) \to C([0,r])$.

Answer. As a function of x the integral is Lipschitz continuous if it exists as the integral of a bounded function. The integral exists for every $x \in [0,r]$ because $s \to 1 + f(s)^2$ is continuous on $[0,r]$).

c) Voor $f \in C([0,r])$ we write

$$|f|_r = \max_{x\in[0,r]} |f(x)|$$

for the maximum norm of f. Show that

$$|\Phi(f) - \Phi(g)|_r \le r\,(|f|_r + |g|_r)\,|f-g|_r$$

for every $f, g \in C([0,r])$.

Answer. Use

$$(\Phi(f) - \Phi(g))(x) = (\Phi(f))(x) - (\Phi(g))(x) = \int_0^x (1+f(s)^2)\,ds - \int_0^x (1+g(s)^2)\,ds$$

$$= \int_0^x (f(s)^2 - g(s)^2)\,ds.$$

We have

$$f(s)^2 - g(s)^2 = (f(s)-g(s))(f(s)+g(s)),$$

so

$$|(1+f(s)^2) - (1+g(s)^2)| \le |f(s)-g(s)|\,|f(s)+g(s)| \le |f-g|_r|f+g|_r \le (|f|_r + |g|_r)|f-g|_r,$$

whence

$$|\Phi(f) - \Phi(g))(x)| = |\int_0^x (f(s)^2 - g(s)^2)\,ds| \le r\,(|f|_r + |g|_r)|f-g|_r$$

for all $x \in [0,r]$, which proves the estimate.

d) Let $r, R > 0$ and

$$A = A_{rR} = \{f \in C([0,r]) : |f|_r \le R\}.$$

Show that

$$|\Phi(f)|_r \le r\,(1+R^2)$$

for every $f \in A$.

Answer.

$$|(\Phi(f))(x)| = |\int_0^x (1+f(s)^2)\,ds| \le \int_0^x |1+f(s)^2|)\,ds = \int_0^x (1+f(s)^2\,ds$$

$$\le \int_0^r (1+\underbrace{f(s)^2}_{\le |f|_r^2})\,ds \le r(1+|f|_r^2)$$

for all $x \in [0,r]$.

e) Show there exist $r > 0$ and $R > 0$ such that Φ is a contraction on A.

Answer. To have Φ map A to A we need $r(1+R^2) \le R$, to have Φ contractive on A we need $2Rr < 1$. Take your pick of $R > 0$ and $r > 0$ for which these both hold.

f) Bonus (2 points): describe the set of all $r > 0$ and $R > 0$ for which Φ is a contraction on A.