**Vrije Universiteit Amsterdam** — **May 29 2020**
**Mathematical Analysis, open course notes take home exam** — **First examiner: Joost Hulshof**
**3 hours** — **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers (refer to theorems used). You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed, except the course notes, use them!

Your grade will be $1 + \frac{T}{4}$, $T$ your total score, maximal $T = 36$.

**Problem 1.** Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \frac{x}{1+x}$$

for all $x \neq -1$ and $F(-1) = 1$. This $F$ can next earn you

**a + b + c + d = 1 + 2 + 3 + 3 = 9 points.**

a) *Sketch the graph of $F$.* Make sure you got it right for $x \geq 0$. Which interval is $\{F(x) : x \geq 0\}$?

b) Factorise $F(x) - F(y)$ and *prove* that $F$ is Lipschitz continuous on $[0, \infty)$ with Lipschitz constant 1.

Consider the integral equation

$$f(x) = 1 + \int_0^x \frac{f(s)}{(1+s)^2(1+f(s))}\, ds \quad \text{posed for all} \quad x \in [0,1] \tag{1}$$

and denote the right hand side of (1) by $(\Phi(f))(x)$. This defines a map $\Phi : A \to A$, where

$$A = \{f \in C([0,1]) : f(x) \geq 0 \text{ for all } x \in [0,1]\}$$

is the subset of nonnegative functions in $C([0,1])$.

c) *Prove that $\Phi$ is a contraction.* Hint: use b), in your estimates you may use that

$$\int_0^1 \frac{1}{(1+s)^2}\, ds < \int_0^\infty \frac{1}{(1+s)^2}\, ds = 1.$$

d) *Explain why it follows* that (1) has a unique *positive* solution $f$.

**Problem 2.** Let $F$ be the function defined in Problem 1. This same $F$ can next earn you another

**a + b + c = 2 + 3 + 3 = 8 points**

a) *Prove* that $F$ is discontinuous in $x = -1$.

b) Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. For every $n \in \mathbb{N}$ we define $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_n(x) = \frac{nx}{1+nx} \quad \text{for all} \quad x \in \mathbb{R}_+.$$

*Prove* that $f_n(x) \to 1$ as $n \to \infty$ for every $x \in \mathbb{R}_+$.

c) *Prove* that the convergence in b) is uniform on $[1, \infty)$.
Hint: estimate $|f_n(x) - 1|$ for all $x \in [1, \infty)$ simultaneously.

**Problem 3.** Consider the differential equation $f''(x) = f(x)$.

a) Use a power series solution of the form

$$a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \cdots$$

*to find an even solution with* $f(0) = 1$. You may guess the expression for $a_{2n}$ from your calculations.

b) The power series that you (should) have found converges for all $x \in \mathbb{R}$. You don't need your answer to a) to continue. The derivative of $f$ is an odd solution denoted by $g$. *Explain in detail why*

$$(f(x))^2 - (g(x))^2$$

*is constant. Which constant?*

**a + b = 2 + 4 = 6 points**

**Problem 4.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x(1 + x).$$

This $f$ can next earn you

**a + b + c + d + e = 3 + 1 + 3 + 3 + 5 = 15 points**

a) The linear approximation of $f(x)$ near $x = 0$ is given by $x$. *Verify the epsilon-delta statement for the remainder term* that implies that $f$ is differentiable in $x = 0$ with $f'(0) = 1$.

b) Now consider for $y \in \mathbb{R}$ the equation

$$f(x) = x(1 + x) = y \tag{2}$$

and the *modified* Newton scheme $x_n = x_{n-1} + f'(0)^{-1}(y - f(x_{n-1}))$ to solve $f(x) = y$. *Verify that*

$$x_n = y - x_{n-1}^2. \tag{3}$$

c) Starting from $x_0 = 0$ the scheme (3) defines a sequence $x_n$ that depends on $y$, and thereby a sequence of functions $g_n$ by defining $g_n(y) = x_n$. So $g_0(y) = 0$ and $g_1(y) = y$. *Evaluate* $g_2(y)$ *and* $g_3(y)$.

Next it's about finding an (inverse) function $g$ as the uniform limit of the sequence $g_n$.

d) Following the notation in (3.1) in Chapter 3 and avoiding the Greek letter $\xi$ we write $s_n = x_n - x_{n-1}$, with $s$ for *step*. Now suppose that for some $n \in \mathbb{N}$ it holds that

$$|x_{n-1}| \leq \frac{1}{4} \quad \text{and} \quad |x_n| \leq \frac{1}{4}.$$

*Use (3) to prove that then*

$$|s_{n+1}| \leq \frac{1}{2}|s_n|. \tag{4}$$

This allows to continue the story-line in Chapter 3 for all $y \in [-\frac{1}{8}, \frac{1}{8}]$ simultaneously.

e) Use d) to show that $g_n$ is a uniform Cauchy sequence in $C([-\frac{1}{8}, \frac{1}{8}])$, see Definition 4.2. Hint: recall that $x_n = g_n(y)$ and *show first that*

$$|y| \leq \frac{1}{8} \implies |x_n| \leq 2|y| \leq \frac{1}{4} \quad \text{for all} \quad n \in \mathbb{N}.$$