Vrije Universiteit Amsterdam | **May 29 2020**
Mathematical Analysis, open course notes take home exam | **First examiner: Joost Hulshof**
3 hours | **Second examiner: Bob Rink**

Write the calculations and arguments that lead to your answers. *Motivate* your answers (refer to theorems used). You can use *earlier* statements, even if you failed to prove them. Calculators/communication/internet sources *NOT* allowed, except the course notes, use them!

Your grade will be $1 + \frac{T}{4}$, T your total score, maximal $T = 36$.

Problem 1. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \frac{x}{1+x}$$

for all $x \neq -1$ and $F(-1) = 1$. This F can next earn you

a + b + c + d = 1 + 2 + 3 + 3 = 9 points.

a) *Sketch the graph of* F. Make sure you got it right for $x \geq 0$. Which interval is $\{F(x) : x \geq 0\}$?

b) Factorise $F(x) - F(y)$ and *prove* that F is Lipschitz continuous on $[0, \infty)$ with Lipschitz constant 1.

Consider the integral equation

$$f(x) = 1 + \int_0^x \frac{f(s)}{(1+s)^2(1+f(s))}\, ds \quad \text{posed for all} \quad x \in [0,1] \tag{1}$$

and denote the right hand side of (1) by $(\Phi(f))(x)$. This defines a map $\Phi : A \to A$, where

$$A = \{f \in C([0,1]) : f(x) \geq 0 \text{ for all } x \in [0,1]\}$$

is the subset of nonnegative functions in $C([0,1])$.

c) *Prove that* Φ *is a contraction.* Hint: use b), in your estimates you may use that

$$\int_0^1 \frac{1}{(1+s)^2}\, ds < \int_0^\infty \frac{1}{(1+s)^2}\, ds = 1.$$

d) *Explain why it follows* that (1) has a unique *positive* solution f.

Answers

a) Near $x = 0$ the graph looks like $y = x$, for $|x|$ large like $y = 1$, and $\{F(x) : x \geq 0\} = [0, 1)$.

b) For $x, y \geq 0$ we have

$$F(x) - F(y) = \frac{x}{1+x} - \frac{y}{1+y} = \frac{x(1+y) - (1+x)y}{(1+x)(1+y)} = \frac{x-y}{(1+x)(1+y)},$$

with denominator at least 1 for $x, y \geq 0$, so $|F(x) - F(y)| \leq |x - y|$.

c) Let $f \in A$. Then

$$(\Phi(f))(x) = 1 + \int_0^x \frac{f(s)}{(1+s)^2(1+f(s))}\, ds$$

exists for every $x \in [0,1]$ as one plus the nonnegative integral of a continuous nonnegative function.

As a function of x the new function $\Phi(f) : [0,1] \to [1,\infty)$ is continuous because $\int_0^x \phi$ is (Lipschitz) continuous in x for every integrable $\phi : [0,1] \to \mathbb{R}$. Thus Φ maps A to A.

To see if Φ is a contraction let $f, g \in A$ and write

$$(\Phi(f) - \Phi(g))(x) = (\Phi(f))(x) - (\Phi(g))(x) =$$

$$1 + \int_0^x \frac{f(s)}{(1+s)^2(1+f(s))}\, ds - 1 - \int_0^x \frac{g(s)}{(1+s)^2(1+g(s))}\, ds =$$

$$\int_0^x \frac{1}{(1+s)^2} \underbrace{\left(\frac{f(s)}{1+f(s)} - \frac{g(s)}{1+g(s)} \right)}_{F(f(s))-F(g(s))} ds.$$

Using b) we estimate

$$|(\Phi(f) - \Phi(g))(x)| \le \int_0^x \frac{1}{(1+s)^2} |F(f(s)) - F(g(s))|\, ds$$

$$\le \int_0^x \frac{1}{(1+s)^2} |f(s) - g(s)|\, ds \le \int_0^x \frac{1}{(1+s)^2}\, ds\, ||f - g||_{max}$$

for all $x \in [0,1]$, so

$$||\Phi(f) - \Phi(g)||_{max} \le \underbrace{\int_0^1 \frac{1}{(1+s)^2}\, ds}_{=\frac{1}{2}}\, ||f - g||_{max}.$$

This holds for all f, g in A, so indeed $\Phi : A \to A$ is a contraction, with contraction factor $\frac{1}{2}$.

d) By definition of Φ the nonnegative solutions of (1) are precisely the fixed points of $\Phi : A \to A$. The set A is closed in $C([0,1])$, $C([0,1])$ is complete with metric defined by $d(f,g) = ||f - g||_{max}$. So Φ has a unique fixed point in A, and thereby there exists a unique solution in A.

Problem 2. Let F be the function defined in Problem 1. This same F can next earn you another

a + b + c = 2 + 3 + 3 = 8 points

a) *Prove* that F is discontinuous in $x = -1$.

b) Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. For every $n \in \mathbb{N}$ we define $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_n(x) = \frac{nx}{1+nx} \quad \text{for all} \quad x \in \mathbb{R}_+.$$

Prove that $f_n(x) \to 1$ as $n \to \infty$ for every $x \in \mathbb{R}_+$.

c) *Prove* that the convergence in b) is uniform on $[1,\infty)$.
Hint: estimate $|f_n(x) - 1|$ for all $x \in [1,\infty)$ simultaneously.

Answers

a) For $x \ne -1$ we have

$$F(x) - F(-1) = \frac{x}{1+x} - 1 = -\frac{1}{1+x}$$

which has a denominator that is small for x close to -1. For instance, we have

$$|x+1| < \frac{1}{2} \implies \frac{1}{|x+1|} > 2.$$

It follows that the epsilon-delta statement for continuity in $x = -1$ fails with $\varepsilon = 2$, because for every $\delta > 0$ we can choose x with $|x - -1| < \delta$ as well as $|x - -1| < \frac{1}{2}$, and for such x the inequality $|F(x) - F(-1)| < 2$ fails.

b) Fix $x > 0$. The limit 1 is given, so let $\varepsilon > 0$. We have to prove the existence of an $N \in \mathbb{N}$ such that

$$|f_n(x) - 1| = |\frac{nx}{1+nx} - 1| = \frac{1}{1+nx} < \varepsilon$$

for all $n \geq N$. The desired inequality is equivalent to

$$1 + nx > \frac{1}{\varepsilon} \iff n > \frac{1-\varepsilon}{\varepsilon x}.$$

If $\varepsilon \geq 1$ this always holds, and we can take $N = 1$. For $\varepsilon < 1$ choose $N \in \mathbb{N}$ such that

$$N > \frac{1-\varepsilon}{\varepsilon x}.$$

Then

$$n \geq N > \frac{1-\varepsilon}{\varepsilon x} \quad \text{and thereby} \quad |f_n(x) - 1| < \varepsilon$$

for all $n \geq N$. This completes the proof the way I did the first examples in the course (Exercise 2.32). But quicker is

$$|f_n(x) - 1| = |\frac{nx}{1+nx} - 1| = \frac{1}{1+nx} < \frac{1}{nx} < \varepsilon,$$

and taking

$$N > \frac{1}{\varepsilon x}.$$

c) For $x \geq 1$ we can choose

$$N > \frac{1-\varepsilon}{\varepsilon x}$$

as in b) for all $x \geq 1$ simultaneously, by choosing

$$N > \frac{1-\varepsilon}{\varepsilon}.$$

Then

$$\begin{matrix} n \geq N \\ x \geq 1 \end{matrix} \implies n \geq N \geq \frac{1-\varepsilon}{\varepsilon} \geq \frac{1-\varepsilon}{\varepsilon x},$$

whence

$$n > \frac{1-\varepsilon}{\varepsilon x},$$

equivalent to the desired inequality $|f_n(x) - 1| < \varepsilon$ as before. This completes the proof. Here too we might have taken the quicker approach choosing $N > \frac{1}{\varepsilon}$, as in b).

Problem 3. Consider the differential equation $f''(x) = f(x)$.

a) Use a power series solution of the form

$$a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + a_8 x^8 + \cdots$$

to find an even solution with $f(0) = 1$. You may guess the expression for a_{2n} from your calculations.

b) The power series that you (should) have found converges for all $x \in \mathbb{R}$. You don't need your answer to a) to continue. The derivative of f is an odd solution denoted by g. *Explain in detail why*

$$(f(x))^2 - (g(x))^2$$

is constant. Which constant?

a + b = 2 + 4 = 6 points

Answers

a) Write

$$f(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \cdots,$$
$$f'(x) = 2a_2x + 4a_4x^3 + 6a_6x^5 + 8a_8x^7 + \cdots,$$
$$f''(x) = 2a_2 + 3\cdot 4a_4x^2 + 5\cdot 6a_6x^4 + 7\cdot 8a_8x^6 + \cdots.$$

Use $f(0) = 1$ conclude $a_0 = 1$ and then $f''(x) = f(x)$ to choose $a_2, a_4, \dots$ with

$$2a_2 = a_0 = 1,\ 3\cdot 4a_4 = a_2,\ 5\cdot 6a_6 = a_4, \dots, \text{ whence } \quad a_2 = \frac{1}{2}, \quad a_4 = \frac{1}{4\cdot 3\cdot 2}, \quad a_6 = \frac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2},$$

and recognise the general expression $a_{2n} = \frac{1}{(2n)!}$, consistent also with a_0 and a_2.

b) It's given that the power series are valid for all x. Define $g = f'$. Then $g' = f'' = f$ so by the chain rule the derivative of $f^2 - g^2$ is equal to $2ff' - 2gg' = 2fg - 2gf'' = 2fg - 2gf = 0$ on the whole of $\mathbb{R}$. The mean value theorem now implies that $f(a)^2 - g(a)^2 = f(b)^2 - g(b)^2$ for all $a, b \in \mathbb{R}$, and thus that

$$f(x)^2 - g(x)^2 = f(1)^2 - g(0)^2 = 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x(1 + x).$$

This f can next earn you

a + b + c + d + e = 3 + 1 + 3 + 3 + 5 = 15 points

a) The linear approximation of $f(x)$ near $x = 0$ is given by x. *Verify the epsilon-delta statement for the remainder term* that implies that f is differentiable in $x = 0$ with $f'(0) = 1$.

b) Now consider for $y \in \mathbb{R}$ the equation

$$f(x) = x(1 + x) = y \tag{2}$$

and the *modified* Newton scheme $x_n = x_{n-1} + f'(0)^{-1}(y - f(x_{n-1}))$ to solve $f(x) = y$. *Verify that*

$$x_n = y - x_{n-1}^2. \tag{3}$$

c) Starting from $x_0 = 0$ the scheme (3) defines a sequence x_n that depends on y, and thereby a sequence of functions g_n by defining $g_n(y) = x_n$. So $g_0(y) = 0$ and $g_1(y) = y$. *Evaluate $g_2(y)$ and $g_3(y)$.*

Next it's about finding an (inverse) function g as the uniform limit of the sequence g_n.

d) Following the notation in (3.1) in Chapter 3 and avoiding the Greek letter ξ we write $s_n = x_n - x_{n-1}$, with s for *step*. Now suppose that for some $n \in \mathbb{N}$ it holds that

$$|x_{n-1}| \leq \frac{1}{4} \quad \text{and} \quad |x_n| \leq \frac{1}{4}.$$

Use (3) to prove that then

$$|s_{n+1}| \leq \frac{1}{2}|s_n|. \tag{4}$$

This allows to continue the story-line in Chapter 3 for all $y \in [-\frac{1}{8}, \frac{1}{8}]$ simultaneously.

e) Use d) to show that g_n is a uniform Cauchy sequence in $C([-\frac{1}{8}, \frac{1}{8}])$, see Definition 4.2. Hint: recall that $x_n = g_n(y)$ and *show first that*

$$|y| \leq \frac{1}{8} \implies |x_n| \leq 2|y| \leq \frac{1}{4} \quad \text{for all} \quad n \in \mathbb{N}.$$

Answers

a) Since $f(x) = f(0) + 1\,x + R(x)$ with $|R(x)| = |x|^2 = |x|\,|x| < \varepsilon|x|$ if $0 < |x| < \varepsilon$ it follows that for all $\varepsilon > 0$ there exists $\delta > 0$, namely $\delta = \varepsilon$, such that $0 < |x| < \delta = \varepsilon$ implies $|R(x)| < \varepsilon|x|$. Thus f is differentiable in $x = 0$ with $f'(0) = 1$.

b) The scheme is of the form $x_n = F(x_n)$ with $F(x) = x + f'(0)^{-1}(y - f(x))$. For the given f this gives $F(x) = x + 1^{-1}(y - x(1 + x)) = x + y - x - x^2 = y - x^2$, so $x_n = y - x_{n-1}^2$.

c) With $x_1 = g_1(y) = y$ we have $g_2(y) = x_2 = y - y^2$ and then $g_3(y) = x_2 = y - (y - y^2)^2$.

d) We have

$$s_{n+1} = x_{n+1} - x_n = y - x_n^2 - y + x_{n-1}^2 = -(x_n + x_{n-1})(x_n - x_{n-1}) = (x_n + x_{n-1})s_n,$$

and thereby

$$|s_{n+1}| \le |x_n + x_{n-1}|\,|s_n| \le (|x_n| + |x_{n-1}|)\,|s_n| \le (\frac{1}{4} + \frac{1}{4})|s_n| = \frac{1}{2}\,s_n.$$

Note that just like x_n the s_n are y-dependent.

e) We have to get $|x_m - x_n| = |g_m(y) - g_n(y)| < \varepsilon$ for all y with $|y| \le \frac{1}{8}$ simultaneously, provided $m > n \ge N$, $N \in \mathbb{N}$ to be found. Following the hint we start from

$$|x_1| = |s_1| = |y| \le 2|y| \le \frac{2}{8} = \frac{1}{4}$$

to get

$$|x_2| \le |x_1| + |s_2| \le |s_1| + \frac{1}{2}\,|s_1| = (1 + \frac{1}{2})|s_1| \le 2|y| \le \frac{1}{4},$$

whence

$$|x_3| \le |x_2| + |s_3| \le (1 + \frac{1}{2})|s_1| + \frac{1}{2}|s_2| \le (1 + \frac{1}{2} + \frac{1}{4})|s_1| \le 2|y| \le \frac{1}{4}.$$

In the next step we have

$$|x_4| \le |x_3| + |s_4| \le (1 + \frac{1}{2} + \frac{1}{4})|s_1| + |s_4| \le (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8})|s_1| \le 2|y| \le \frac{1}{4},$$

because $|s_4| \le \frac{1}{2}|s_3| \le \frac{1}{4}|s_2| \le \frac{1}{8}|s_1|$.

And so on. We conclude that all $|x_n| = |g_n(y)|$ are bounded by $\frac{1}{4}$, whence

$$|s_{n+1}| \le \frac{1}{2^n}|s_1| = \frac{1}{2^n}|y| \le \frac{1}{2^{n+3}}$$

for all y with $|y| \le \frac{1}{8}$. But then we have for all such y and $m > n \ge N$ that

$$|g_n(y) - g_m(y)| = |x_m - x_n| \le |s_{n+1}| + \cdots + |s_m| \le |s_{n+1}|\,\underbrace{(1 + \frac{1}{2} + \frac{1}{4} + \cdots)}_{\text{finitely many terms}} \le 2|s_{n+1}| \le \frac{1}{2^{n+2}} \le \frac{1}{2^{N+2}}.$$

Choosing N so large that $2^{N+2} > \frac{1}{\varepsilon}$ the proof that g_n is uniformly Cauchy on the y-interval $[-\frac{1}{8}, \frac{1}{8}]$ is now complete.