Vrije Universiteit Amsterdam | **July 4 2019, 12.00-14.45**
Mathematical Analysis, Retake | **Lecturer: Joost Hulshof**

Write the calculations and arguments that lead to your answers.
Motivate your answers (mention theorems used).
You (may need and) can use *earlier* statements, even if you failed to prove them.
NO calculators/computers/phones allowed.

For this exam you have 2 hours and 45 minutes. You can score 30 points.
Your grade will be $1 + \frac{1}{3} \times$ your total score, with a maximum of 10.

Problem 1. (3 *points*) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \cos \frac{1}{x} & \text{for } x \neq 0 \end{cases}.$$

Prove that f is Riemann integrable on $[0, 1]$.

Problem 2. (3 *points*) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ x(42 + |x|^{\frac{1}{41}} \sin \frac{1}{x^{43}}) & \text{for } x \neq 0 \end{cases}$$

Prove that f is differentiable in $x = 0$. Hint: what would the value of $f'(0)$ be?

Problem 3. Let the sequence of functions $f_n : [0, 1] \to [0, 1]$ be defined by $f_n(x) = x^n$ for all $x \in [0, 1]$.

a) (2 *points*) Formulate the Archimedean Principle. Use it to show that

$$f_n(x) \to 0$$

as $n \to \infty$ for every $x \in (0, 1)$.

b) (2 *points*) For $n \in \mathbb{N}$ and $a \in \mathbb{R}$ we define $R_n(x, a)$ by

$$f_n(x) = x^n = a^n + na^{n-1}(x - a) + R_n(x, a)$$

for $x \in \mathbb{R}$. Compute $R_3(x, a)$ and prove that f_3 is differentiable in $x = a$.

c) (2 *points*) The functions f_n belong to the complete metric space $C([0, 1])$ in which the distance between f_n and f_m is defined by

$$d(f_m, f_n) = \max_{x \in [0,1]} |f_m(x) - f_n(x)|.$$

Let $m > n$ and

$$x_{mn} = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}.$$

Explain why

$$d(f_m, f_n) = f_n(x_{mn}) - f_m(x_{mn}).$$

Hint: from (b) you know what derivative of $f_m - f_n$ is.

d) (2 *points*) Show that f_n is not a Cauchy sequence in $C([0, 1])$. Hint: determine $d(f_{2n}, f_n)$.

Problem 4. In the course we defined $\sin x$ and $\cos x$ by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

for all $x \in \mathbb{R}$. Use this definition to answer the four questions below.

a) (1 *point*) Explain why $\cos 1 > 0$. Hint: what is the sign of $1 - \frac{1}{2!}$, $\frac{1}{4!} - \frac{1}{6!}$, ... ?

b) (1 *point*) Explain why $\cos 1 < 1$. Hint: what is the sign of $\frac{1}{2!} - \frac{1}{4!}$, ... ?

c) (1 *point*) Explain why $0 < \sin x < x$ for every $x \in (0, 1]$. Hint: what is the sign of ...?

d) (1 *point*) Explain why $\sin' = \cos$ and $\cos' = -\sin$.

Problem 5. In this exercise you can use what is stated in Exercise 4 about the functions cos and sin.

a) (1 *point*) Make a sketch of $y = x$ and $y = \cos x$ in the x, y-plane with $0 \leq x, y \leq 1$.

b) (1 *point*) Prove that

$$\cos : [0, \cos 1] \to [0, \cos 1]$$

is a contraction. Hint: use the mean value theorem.

c) (2 *points*) Explain why $x = \cos x$ has a unique solution in $[0, 1]$.

d) (2 *points*) Prove that the integral equation

$$f(x) = \int_0^x \cos f(s)\, ds \quad \text{for all} \quad x \in [0, 1]$$

has a unique solution in $C([0, 1])$.

Problem 6. Let the continuous function $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^{\frac{2}{3}}$. For $n \in \mathbb{N}$ we define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x_j) = f(x_j) \quad \text{for} \quad x_j = \frac{j}{n}, \quad j = 0, 1, 2, \ldots, n,$$

and by f_n being linear on every interval $I_j = [\frac{j-1}{n}, \frac{j}{n}]$. The functions f_n are all Lipschitz continuous.

a) (1 *point*) What is the Lipschitz constant of f_2? Hint: make a sketch of the graph of f_2.

b) (1 *point*) State a theorem which implies that f is uniformly continuous.

c) (2 *points*) For $\varepsilon > 0$ let $\delta > 0$ be given by the definition of uniform continuity of f, i.e.

$$\forall_{x,y \in [0,1]} : |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Let $n \in \mathbb{N}$ satisfy $n > \frac{1}{\delta}$. Prove that

$$|f_n(x) - f(x)| < 2\varepsilon$$

for all $x \in [0, 1]$. Hint: given $x \in [0, 1]$ use the inequality

$$|f_n(x) - f(x)| \leq |f_n(x) - f(x_j)| + |f(x_j) - f(x)|$$

and choose j such that $x \in I_j$.

d) (2 *points*) Use (c) to show that $f_n \to f$ uniformly on $[0, 1]$.