VU Amsterdam	Mathematical Analysis (XB_0009)
Faculty of Sciences	Solutions to Resit Exam 2019
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Solution to Problem 1.

The given function is bounded on $[0,1]$. Indeed, $|f(x)| \leq 1$ for all $x \in [0,1]$. Moreover, it is discontinuous only at $x = 0$. From the lectures we know that if $f\colon [a,b] \to \mathbb{R}$ is a bounded function with a finite number of discontinuities,then f is integrable. This finishes the proof.

Solution to Problem 2.

We have

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} 42 + |x|^{\frac{1}{41}} \sin(\tfrac{1}{x^{43}}) = 42 + \lim_{x\to 0} |x|^{\frac{1}{41}} \sin(\tfrac{1}{x^{43}}) = 42 + 0 = 42\,.$$

To compute the last limit, we used the sandwich theorem. Indeed, $-|x|^{\frac{1}{41}} \leq |x|^{\frac{1}{41}} \sin(\frac{1}{x^{43}} \leq |x|^{\frac{1}{41}}$ and we know that $\lim_{x\to 0} |x|^{\frac{1}{41}} = 0$.

Solution to Problem 3.

a) The Archimedean property says that for all $M \in \mathbb{R}$ there is $N \in \mathbb{N}$ such that $N > M$. If $x \in (0,1)$, let $y := 1-x$. Then $x = 1 - y = \frac{1}{1+\frac{y}{1-y}}$. Call $z := \frac{y}{1-y} \in (0,\infty)$ so that $x = \frac{1}{1+z}$. Let $\varepsilon > 0$. Consider $n \geq N$ where N is to be chosen below. Then,

$$0 \leq x^n = \frac{1}{(1+z)^n} \leq \frac{1}{1+nz} \leq \frac{1}{1+Nz}\,.$$

This last quantity is less than $\varepsilon > 0$ if $N > M := \frac{1-\frac{1}{\varepsilon}}{z}$. Such an N is obtained by the Archimedean property. By definition of convergence, we see that $\lim_{n\to\infty} x^n = 0$ for all $x \in (0,1)$.

b) We have

$$\begin{aligned} x^3 - a^3 - 3a^2(x-a) = (x^2+ax+a^2)(x-a) - 3a^2(x-a) &= (x^2+ax-2a^2)(x-a) \\ &= (x+2a)(x-a)(x-a)\,. \end{aligned}$$

So $R_3(a,x) = \rho(x)(x-a)$ where $\rho(x) = (x+2a)(x-a)$. Now ρ is a continuous function with $\rho(a) = 0$. We deduce that f is differentiable at $x = a$ with derivative equal to $3a^2$.

c) If $m > n$, then $|f_m(x) - f_n(x)| = |x^m - x^n| = x^n - x^m := g(x)$. The function $g\colon [0,1] \to \mathbb{R}$ is differentiable and non-negative. To compute its maximum we consider the derivative $g'(x) = nx^{n-1} - mx^{m-1}$. Suppose that $g'(x) = 0$ and $x \in (0,1)$. Then $x^{m-1-(n-1)} = \frac{n}{m}$, which means that $x = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$. Since $g(0) = 0$ and $g(1) = 0$, we deduce that g has a maximum at $x_{mn} := \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$. Therefore, $d(f_m, f_n) = f_n(x_{mn}) - f_m(x_{mn})$.

d) If (f_n) were a Cauchy sequence, then the sequence of real numbers $d(f_{2n}, f_n)$ converges to 0. Indeed, for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(f_m, f_n) < \varepsilon$ for all $n, m \geq N$. In particular, taking $m = 2n$, we get $|d(f_{2n}, f_n)| < \varepsilon$ for all $n \geq N$. Now $x_{2nn} = \frac{1}{2^{\frac{1}{n}}}$. Therefore, $d(f_{2n}, f_n) = f_n(x_{2nn}) - f_{2n}(x_{2nn}) = \frac{1}{2} - \frac{1}{2^2} = \frac{1}{4}$ which is not converging to 0.

Solution to Problem 4.

This problem is not relevant for the 2023 course.

Solution to Problem 5.

a),b),c) The solution can be found in Problem 2 of the 2021 final exam.

d) Consider the set $A := \{f \in C[0,1] \;:\; 0 \leq f(x) \leq 1,\ \forall x \in [0,1]\}$. This set is non-empty and closed with respect to the uniform metric d. If $f \in A$, then $0 \leq \cos(f(s)) \leq 1$ for all $s \in [0,1]$ and therefore

$$0 \leq \int_0^x \cos(f(s))\mathrm{d}s \leq \int_0^x 1\mathrm{d}s = x \leq 1\,.$$

Thus, the map $\Phi\colon A \to A$ is well-defined. By a), we know that there is $r \in (0,1]$ such that $|\cos x - \cos y| \leq r|x-y|$ for all $x, y \in [0,1]$. With this information we can show that the map Φ is a contraction with respect to the uniform metric d. Indeed, for all $f, g \in A$ and all $x \in [0,1]$ we have

$$\begin{aligned} |\Phi(f)(x) - \Phi(g)(x)| \leq \int_0^x |\cos(f(s)) - \cos(g(s))|\mathrm{d}s &\leq \int_0^x r|f(s)-g(s)|\mathrm{d}s \\ &\leq \int_0^x rd(f,g)\mathrm{d}s \\ &= xrd(f,g) \\ &\leq rd(f,g) \end{aligned}$$

Thus, $d(T(f), T(g)) \leq rd(f,g)$ showing that T is a contraction. Since A is a non-empty closed subset of the complete metric space $C[0,1]$, the Banach contraction principle implies that Φ has a unique fixed point f. This f satisfies the desired equation.

Solution to Problem 6.

a) The function f_2 in the interval $[0, \frac{1}{2}]$ has slope $L_1 = \frac{f(\frac{1}{2})-f(0)}{\frac{1}{2}} = \frac{2}{4^{\frac{1}{3}}}$. The function f_2 in the interval $[\frac{1}{2}, 1]$ has slope $L_2 = \frac{f(1)-f(\frac{1}{2})}{\frac{1}{2}} = \frac{2}{4^{\frac{1}{3}}}(4^{\frac{1}{3}} - 1) < \frac{2}{4^{\frac{1}{3}}} = L_1$. Therefore, the smallest possible Lipschitz constant r_2 for f_2 in the whole interval $[0,1]$ is equal to L_1.

b) Since f is continuous on a compact subset of $\mathbb{R}$, it is uniformly continuous.

c) Let $x \in [0,1]$. Take j such that $x \in I_j$. Then,

$$|f_n(x) - f(x)| \leq |f_n(x) - f(x_j)| + |f(x_j) - f(x)| = |f_n(x) - f_n(x_j)| + |f(x_j) - f(x)|\,,$$

where we used that $f_n(x_j) = f(x_j)$. Now, $f_n(x_j)$ lies between $f_n(x_{j-1})$ and $f_n(x_j)$ since f_n is linear on the interval I_j. Therefore,

$$|f_n(x) - f_n(x_j)| \leq |f_n(x_{j-1}) - f_n(x_j)| = |f(x_{j-1}) - f(x_j)| \, .$$

Thus,

$$|f_n(x) - f(x)| \leq |f(x_{j-1}) - f(x_j)| + |f(x_j) - f(x)| < \varepsilon + \varepsilon \, ,$$

since $|x - x_j| \leq \frac{1}{n} < \delta$ and $|x_{j-1} - x_j| \leq \frac{1}{n} < \delta$.

d) Let $\varepsilon > 0$ and let $\delta > 0$ the suitable response for ε in the definition of uniform continuity of f. Take $N \in \mathbb{N}$ such that $N > \frac{1}{\delta}$. Then for all $n \geq N$, we have $n > \frac{1}{\delta}$ and by c) we conclude that $d(f_n, f) < 2\varepsilon$ where d is the uniform metric. This means that $f_n \to f$ uniformly on $[0, 1]$.