Vrije Universiteit Amsterdam — **May 23, 2019**
Mathematical Analysis, Final Exam — **Lecturer: Joost Hulshof**

Write the calculations and arguments that lead to your answers.
Motivate your answers (mention theorems used).
You can use *earlier* statements, even if you failed to prove them.
NO calculators/computers/phones allowed.

You have 2 hours and 45 minutes. Exercises 1,2,3,5 count for 6 points each, 4 counts for 3 points. If S is your totale score than your grade will be $1+\frac{S}{3}$.

Good Luck! As promised two questions are without epsilons:

Question 1. $(2+1+3)$ In this exercise you will have to use the Banach Contraction Theorem in $C([0,1])$.

a) Show that the function $g:\mathbb{R}\to[0,1]$ defined by

$$g(x)=\frac{1}{1+x^2}$$

is Lipschitz continuous with Lipschitz constant $L=1$. Hint: factorise $g(x)-g(y)$. You may use without proof that

$$-\frac{1}{2}\le\frac{x}{1+x^2}\le\frac{1}{2}.$$

b) Prove that g is uniformly continuous. Specify the choice of $\delta>0$ in the definition for given $\varepsilon>0$.

c) Prove that the integral equation

$$f(x)=\int_0^x\frac{1}{(1+s)(1+f(s)^2)}\,ds\quad\text{for all}\quad x\in[0,1]$$

has a unique solution f in $C([0,1])$.

Question 2. (2+2+2) The differential equation $p''(x)=p(x)$ has a power series solution which is convergent for every $x\in\mathbb{R}$ and satisfies the conditions $p(0)=1$ and $p'(0)=0$.

a) Show that it is of the form

$$p(x)=\sum_{n=0}^{\infty}a_nx^{2n}\qquad\text{and find an expression for } a_n.$$

b) Define $q(x)$ by $q(x)=p'(x)$. Show that the derivative of $p(x)^2-q(x)^2$ is zero for all $x\in\mathbb{R}$.

c) Formulate the theorem that implies that $p(x)^2-q(x)^2$ is constant, and determine the constant.

The epsilons occur in Questions 3,4,5 on the other side, 2- and 3-tricks are allowed.

Question 3. $(1+2+3)$ Let $f:[0,1]\to[-1,1]$ be given by

$$f(x)=\begin{cases}0 & \text{for } x=0\\ \sin\frac{1}{x} & \text{for } x\neq 0\end{cases}.$$ Recall that we write, for a partition $0\leq x_0\leq x_1\leq\cdots\leq x_N=1$,

$$I_k=[x_{k-1},x_k],\quad m_k=\inf_{I_k} f,\quad M_k=\sup_{I_k} f,\quad \overline{S}=\sum_{k=1}^{N} M_k\,(x_k-x_{k-1}),\quad \underline{S}=\sum_{k=1}^{N} m_k\,(x_k-x_{k-1}).$$

a) Let $\varepsilon>0$ and $a\in(0,1]$. Prove that f is Riemann integrable on $[a,1]$, for instance by using a theorem.

b) Use (a) and another theorem to prove the existence of such a partition with $x_0=a$ for which $\overline{S}-\underline{S}<\varepsilon$.

c) Prove that f is Riemann integrable on $[0,1]$. Hint: start with $\varepsilon>0$ and choose $x_0=0<x_1=a<\varepsilon$.

Question 4. (3) Let $f:\mathbb{R}\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}0 & \text{for } x=0\\ x(1+\sqrt{|x|}\sin\frac{1}{x}) & \text{for } x\neq 0\end{cases}$$

Prove that f is differentiable in $x=0$: give the linear approximation of $f(x)$ near $x=0$ and verify the ε-δ statement for the remainder term. Specify $\delta>0$ for given $\varepsilon>0$.

Question 5. $(2+2+2)$ Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\sqrt{x}$. For $n\in\mathbb{N}$ define $f_n:[0,1]\to\mathbb{R}$ by

$$f_n(x_j)=f(x_j)\quad\text{for}\quad x_j=\frac{j}{n},\quad j=0,1,2,\ldots,n,$$

and by f_n being linear on every interval $I_j=[\frac{j-1}{n},\frac{j}{n}]$.

a) Sketch the graph of f_4 and explain why f_4 is Lipschitz continuous with Lipschitz constant $\frac{1}{2}$.

b) For $\varepsilon>0$ let $\delta>0$ be given by the definition of uniform continuity of f, i.e.

$$\forall_{x,y\in[0,1]}\,:\,|x-y|<\delta\implies|f(x)-f(y)|<\varepsilon,$$

and let $n\in\mathbb{N}$ satisfy $n>\frac{1}{\delta}$. Prove that

$$|f_n(x)-f(x)|<2\varepsilon$$

for all $x\in[0,1]$. Hint: given $x\in[0,1]$ use the inequality

$$|f_n(x)-f(x)|\leq|f_n(x)-f(x_j)|+|f(x_j)-f(x)|,$$

choose j such that $x\in I_j$, and then use the definition of f_n to show that both terms are less than ε.

c) Use (b) to show that $f_n\to f$ uniformly on $[0,1]$.