Vrije Universiteit Amsterdam **May 23, 2019**
Mathematical Analysis, Final Exam **Lecturer: Joost Hulshof**
Exercises 1,2,3,5 count for 6 points each, 4 counts for 3 points, 3 points for free. Divide by 3.

Question 1. $(2+1+3)$ In this exercise you will have to use the Banach Contraction Theorem in $C([0,1])$.

a) Show that the function $g : \mathbb{R} \to [0,1]$ defined by

$$g(x) = \frac{1}{1+x^2}$$

is Lipschitz continuous with Lipschitz constant $L = 1$. Hint: factorise $g(x) - g(y)$. You may use without proof that

$$-\frac{1}{2} \le \frac{x}{1+x^2} \le \frac{1}{2}.$$

We have

$$g(x) - g(y) = \frac{1}{1+x^2} - \frac{1}{1+y^2} = \frac{y^2 - x^2}{(1+x^2)(1+y^2)} = \frac{x+y}{(1+x^2)(1+y^2)}\,(y-x),$$

and

$$\left|\frac{x+y}{(1+x^2)(1+y^2)}\right| \le \frac{|x|}{(1+x^2)(1+y^2)} + \frac{|y|}{(1+x^2)(1+y^2)} \le \frac{|x|}{1+x^2} + \frac{|y|}{1+y^2} \le \frac{1}{2} + \frac{1}{2},$$

so $|g(x) - g(y)| \le |x-y|$ *for all* $x, y \in \mathbb{R}$.

b) Prove that g is uniformly continuous. Specify the choice of $\delta > 0$ in the definition for given $\varepsilon > 0$.

Let $\varepsilon > 0$. *Use the same estimate or the statement in (a). Since* $|g(x) - g(y)| \le |x-y|$ *the choice* $\delta = \varepsilon$ *does the job:* $x - y| < \delta = \varepsilon$ *implies that* $|g(x) - g(y)| \le |x-y| < \varepsilon$.

c) Prove that the integral equation

$$f(x) = \int_0^x \frac{1}{(1+s)(1+f(s)^2)}\,ds \quad \text{for all} \quad x \in [0,1]$$

has a unique solution f in $C([0,1])$. *Use the right hand side to define a new function, say* F, *by*

$$F(x) = \int_0^x \frac{1}{(1+s)(1+f(s)^2)}\,ds \quad \textit{for all} \quad x \in [0,1].$$

Since the integrand is bounded, by 1 *in fact, we have, using the "triangle inequality" for integrals,*

$$|F(x) - F(y)| = \left|\int_y^x \frac{1}{(1+s)(1+f(s)^2)}\,ds\right| \le \left|\int_y^x \left|\frac{1}{(1+s)(1+f(s)^2)}\right| ds\right| \le \left|\int_y^x 1\,ds\right| = |x-y|$$

so f *is Lipschitz continuous with Lipschitz constant* 1. *In particular* $F \in C([0,1])$ *and the map* Φ *is well defined by*

$$f \xrightarrow{\Phi} F \quad \textit{from } C([0,1]) \textit{ to } C([0,1]).$$

To see if Φ *is a contraction we look at* $F = \Phi(f)$ *and* $G = \Phi(g)$. *The difference* $F - G$ *is defined by* $(F-G)(x) = F(x) - G(x) =$

$$\int_0^x \frac{1}{(1+s)(1+f(s)^2)}\,ds - \int_0^x \frac{1}{(1+s)(1+g(s)^2)}\,ds = \int_0^x \frac{1}{1+s}\left(\frac{1}{1+f(s)^2} - \frac{1}{1+g(s)^2}\right) ds,$$

so by (a) and the "triangle inequality" for integrals again

$$|(F-G)(x)| \le \int_0^x \frac{1}{1+s}\,|f(s) - g(s)|\,ds \le \int_0^x \frac{1}{1+s}\,d(f,g)\,ds \le \int_0^1 \frac{1}{1+s}\,ds\,d(f,g) = \ln 2\,d(f,g)$$

for all $x \in [0,1]$ *so* $d(\Phi(f), \Phi(g)) \le \ln 2\, d(f,g)$ *for all* $f, g \in C([0,1])$. *Thus* Φ *is a contraction on the complete metric space* $C([0,1])$ *and therefore* $f = \Phi(f)$ *has a unique solution in* $C([0,1])$.

Question 2. (2+2+2) The differential equation $p''(x) = p(x)$ has a power series solution which is convergent for every $x \in \mathbb{R}$ and satisfies the conditions $p(0) = 1$ and $p'(0) = 0$.

a) Show that it is of the form

$$p(x) = \sum_{n=0}^{\infty} a_n x^{2n} \qquad \text{and find an expression for } a_n.$$

Write

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6 + \cdots,$$

then

$$p'(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 + 5\alpha_5 x^4 + 6\alpha_6 x^5 + 7\alpha_7 x^6 + \cdots,$$

$$p''(x) = 2\alpha_2 + 3\times 2\alpha_3 x + 4\times 3\alpha_4 x^2 + 5\times 4\alpha_5 x^3 + 6\times 5\alpha_6 x^4 + 7\times 6\alpha_7 x^5 + 8\times 7\alpha_8 x^4 + \cdots,$$

so $p''(x) = p(x)$ gives, comparing the coeffcients,

$$2\alpha_2 = \alpha_0, \quad 4\times 3\alpha_4 = \alpha_2, \quad 6\times 5\alpha_6 = \alpha_4, \quad 8\times 7\alpha_8 = \alpha_6,$$

$$2\alpha_3 = \alpha_1, \quad 5\times 4\alpha_5 = \alpha_3, \quad 7\times 6\alpha_7 = \alpha_5, \quad 9\times 8\alpha_9 = \alpha_7,$$

and so on. Since $\alpha_0 = p(0) = 1$ and $\alpha_1 = p'(0) = 0$ it follows that $0 = \alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = \cdots$, and

$$\alpha_2 = \frac{1}{2}, \quad \alpha_4 = \frac{1}{4\times 3}\alpha_2 = \frac{1}{4!}, \quad \alpha_6 = \frac{1}{6\times 5}\alpha_4 = \frac{1}{6!}, \ldots,$$

so

$$p(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad \text{i.e.} \quad a_n = \frac{1}{(2n)!}.$$

b) Define $q(x)$ by $q(x) = p'(x)$. Show that the derivative of $p(x)^2 - q(x)^2$ is zero for all $x \in \mathbb{R}$.

With $q(x) = p'(x)$ it follows from the differential equation that $q' = p'' = p$. The Leibniz rule (which we proved for power series first) gives that the derivative of p^2 is $p'p + pp' = 2pp'$, a special case of the chain rule. You can also directly use that $2pp'$ is the derivative of p^2. Likewise the derivative of q^2 is $2qq' = 2p'p'' = 2p'p$. So p^2 and q^2 have the same derivative and thus the derivative of $p^2 - q^2$ is identically equal to zero.

c) Formulate the theorem that implies that $p(x)^2 - q(x)^2$ is constant, and determine the constant.

The fact that $f(x) = p(x)^2 - q(x)^2$ is constant follows from the differentiability of f in every $\mathbb{R}$ and the mean value theorem applied to f on $[a, b]$. Indeed, the theorem says, for $f \in C([a, b])$ with f differentiable on (a, b), that $f(b) - f(a) = f(\xi)(b - a)$ for some $\xi \in (a, b)$. For the f under consideration it follows that $f(b) - f(a) = 0$ for every a and b with $a < b$. Thus $f(b) = f(0)$ for every $b > 0$ and $f(a) = f(0)$ for every $a < 0$. So $f(x) = f(0) = p(0)^2 - q(0)^2 = 1$ for all $x \in \mathbb{R}$.

Question 3. $(1+2+3)$ Let $f:[0,1]\to[-1,1]$ be given by

$$f(x)=\begin{cases}0 & \text{for } x=0\\ \sin\frac{1}{x} & \text{for } x\neq 0\end{cases}.\quad \text{Recall that we write, for a partition } 0\le x_0\le x_1\le\cdots\le x_N=1,$$

$$I_k=[x_{k-1},x_k],\quad m_k=\inf_{I_k} f,\quad M_k=\sup_{I_k} f,\quad \overline{S}=\sum_{k=1}^{N} M_k\,(x_k-x_{k-1}),\quad \underline{S}=\sum_{k=1}^{N} m_k\,(x_k-x_{k-1}).$$

a) Let $\varepsilon>0$ and $a\in(0,1]$. Prove that f is Riemann integrable on $[a,1]$, for instance by using a theorem.
The function f is continuous on $[a,1]$. Therefore it integrable on $[a,1]$.

b) Use (a) and another theorem to prove the existence of such a partition with $x_0=a$ for which $\overline{S}-\underline{S}<\varepsilon$.
Since f is integrable on $[a,1]$ there exists for every $\varepsilon>0$ a partition $a=x_0\le x_1\le\cdots\le x_N=1$ of $[a,1]$ such that $\overline{S}-\underline{S}<\varepsilon$, by the ε-criterium for integrability of f on $[a,1]$.

c) Prove that f is Riemann integrable on $[0,1]$. Hint[1]: start with $\varepsilon>0$ and choose $x_0=0<x_1=a<\varepsilon$.
To verify the ε-criterion for integrability of f on $[0,1]$ let $\varepsilon>0$. Choose $x_0=0<x_1=a<\varepsilon$. Then use (b) to conclude the existence of a partition $a=x_1\le\cdots\le x_N=1$ of $[a,1]$ with

$$\sum_{k=2}^{N} M_k\,(x_k-x_{k-1})-\sum_{k=2}^{N} m_k\,(x_k-x_{k-1})<\varepsilon.$$

Then

$$\sum_{k=1}^{N} M_k\,(x_k-x_{k-1})-\sum_{k=1}^{N} m_k\,(x_k-x_{k-1})=M_1a+\sum_{k=2}^{N} M_k\,(x_k-x_{k-1})-m_1a-\sum_{k=2}^{N} m_k\,(x_k-x_{k-1})$$

$$<(M_1-m_1)a+\varepsilon\le 2a+\varepsilon<2\varepsilon+\varepsilon=3\varepsilon.$$

in which we used that $-1\le m_1\le M_1\le 1$ whence $M_1-m_1\le 2$. Thus we have for very $\varepsilon>0$ the existence of a partition of $[0,1]$ with $\overline{S}-\underline{S}<3\varepsilon$. By the ε-criterion and a 3-trick the function f is then integrable on $[0,1]$.

Question 4. (3) Let $f:\mathbb{R}\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}0 & \text{for } x=0\\ x(1+\sqrt{|x|}\sin\frac{1}{x}) & \text{for } x\neq 0\end{cases}$$

Prove that f is differentiable in $x=0$: give the linear approximation of $f(x)$ near $x=0$ and verify the ε-δ statement for the remainder term. Specify $\delta>0$ for given $\varepsilon>0$.

We recognise x as the linear approximation for x close to 0. Then $f(x)=x+R(x)$ with

$$R(x)=x\sqrt{|x|}\sin\frac{1}{x}$$

for $x\neq 0$. Let $\varepsilon>0$. Then

$$|R(x)|=\left|x\sqrt{|x|}\sin\frac{1}{x}\right|\le\sqrt{|x|}\,|x|<\varepsilon\,|x|$$

if $x\neq 0$ and $\sqrt{|x|}<\varepsilon$. So choose $\delta>0$ such that $\sqrt{\delta}=\varepsilon$. For $0<|x|<\delta$ it follows that

$$R(x)\le\sqrt{\delta}\,|x|=\varepsilon\,|x|.$$

This completes the proof.

[1] I changed the hint, $<\varepsilon$ instead of $<\frac{\varepsilon}{2}$

Question 5. $(2+2+2)$ Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\sqrt{x}$. For $n\in\mathbb{N}$ define $f_n:[0,1]\to\mathbb{R}$ by

$$f_n(x_j)=f(x_j)\quad\text{for}\quad x_j=\frac{j}{n},\quad j=0,1,2,\ldots,n,$$

and by f_n being linear on every interval $I_j=[\frac{j-1}{n},\frac{j}{n}]$.

a) Sketch the graph of f_4 and explain why f_4 is Lipschitz continuous with Lipschitz constant 2. **Unfortunately there was a mistake in the exam, of course the Lipschitz constant is not $\frac{1}{2}$.**

Draw the piecewise linear curve through $(0,0),(\frac14,\frac12),(\frac12,\sqrt{\frac14}),(\frac34,\sqrt{\frac34}),(0,1)$. *On the first interval* $[0,\frac14]$ *the slope is 2, which is larger than the slope on each of the intervals* $[\frac14,\frac12]$, $[\frac12,\frac34]$, $[\frac34,1]$. *So for* x,y *both in one of the intervals* $I_1=[0,\frac14]$, $I_2=[\frac14,\frac12]$, $I_3=[\frac12,\frac34]$, $I_4=[\frac34,1]$ *we have* $|f(x)-f(y)|\le 2\,|x-y|$. *If not choose, then* x *and* y *are in two different intervals. If these are* I_1 *and* I_2 *we estimate*

$$|f(x)-f(y)|\le|f(x)-f(\frac14)|+|f(\frac14)-f(y)|\le 2\,|x-\frac14|+2\,|\frac14-y|=2\,|x-y|,$$

and likewise if these are I_2 *and* I_3, *or* I_3 *and* I_4, *with* $\frac14$ *replaced by* $\frac12$ *or* $\frac34$. *If these are* $x\in I_1$ *and* $y\in I_3$, *then we choose two intermediate points to conclude*

$$|f(x)-f(y)|\le|f(x)-f(\frac14)|+|f(\frac14)-f(\frac12)|+|f(\frac12)-f(y)|\le 2\,|x-\frac14|+2\,|\frac14-\frac12|+2\,|\frac12-y|=2\,|x-y|.$$

If these are $x\in I_1$ *and* $y\in I_4$ *then*

$$|f(x)-f(y)|\le|f(x)-f(\frac14)|+|f(\frac14)-f(\frac12)|+|f(\frac12)-f(\frac34)|+|f(\frac34)-f(y)|\le\cdots=2\,|x-y|.$$

Likewise in al remaining cases.

b) For $\varepsilon>0$ let $\delta>0$ be given by the definition of uniform continuity of f, i.e.

$$\forall_{x,y\in[0,1]}\ :\ |x-y|<\delta\implies|f(x)-f(y)|<\varepsilon,$$

and let $n\in\mathbb{N}$ satisfy $n>\frac{1}{\delta}$. Prove that

$$|f_n(x)-f(x)|<2\varepsilon$$

for all $x\in[0,1]$. Hint: given $x\in[0,1]$ use the inequality

$$|f_n(x)-f(x)|\le|f_n(x)-f(x_j)|+|f(x_j)-f(x)|,$$

choose j such that $x\in I_j$, and then use the definition of f_n to show that both terms are less than ε.

Let $\varepsilon>0$. *Look at the two terms in*

$$|f_n(x)-f(x)|\le|f_n(x)-f(x_j)|+|f(x_j)-f(x)|.$$

The second is smaller than ε *if* $x\in I_j$ *because then* $0\le x_j-x\le x_j-x_{j-1}|=\frac1n<\delta$. *But also the first is then smaller than* ε, *because* $|f_n(x)-f(x_j)|=|f_n(x)-f_n(x_j)|\le|f_n(x_{j-1})-f_n(x_j)|=|f(x_{j-1})-f(x_j)|$ *and* $|x_j-x_{j-1}|=\frac1n<\delta$. *This proves the claim.*

c) Use (b) to show that $f_n\to f$ uniformly on $[0,1]$.

Let $\varepsilon>0$. *Choose* $N>\frac1\delta$, δ *from the definition of uniform continuity of* f, *and use (b). Then for all* $x\in[0,1]$ *we have*

$$n\ge N\implies n>\frac1\delta\implies|f_n(x)-f(x)|\le 2\varepsilon.$$

This proves

$$\forall_{\varepsilon>0}\,\exists_{N\in\mathbb{N}}\,\forall_{x\in[0,1]}\ :\ n\ge N\implies|f_n(x)-f(x)|\le 2\varepsilon,$$

the definition of uniform convergence with $<\varepsilon$ *replaced by* $<2\varepsilon$. *The 2-trick completes the proof.*