Vrije Universiteit Amsterdam **March 13 2019**
Mathematical Analysis, (rehearsal) Exam 1 **Docent: Joost Hulshof**

Write the calculations and arguments that lead to your answers.
Motivate your answers (mention theorems used).
You can use *earlier* statements, even if you failed to prove them.
NO calculators/computers/phones allowed.

Your grade will be $1 + \frac{\text{your total score}}{\text{total maximal score}}$.

Problem 1. Some basic theory.

a) Formulate the Archimedean Principle.

b) Give the definition of a Cauchy sequence.

c) Give the definition of a convergent sequence.

d) Formulate the Bolzano Weierstrass Theorem.

Problem 2. Consider the sequence x_n indexed by $n \in \mathbb{N}$ defined by

$$x_n = \frac{1}{\sqrt{n}}.$$

a) Prove that x_n is convergent.

b) Prove that 0 is the largest lower bound for the sequence x_n.

Problem 3. For $a \in (0, \frac{1}{4})$ define the sequence x_n by $x_0 = 1$ and

$$x_n = 1 - \frac{a}{x_{n-1}}.$$

a) Use induction to show that $x_n > \frac{1}{2}$ for all $n \in \mathbb{N}$.

b) Use induction to show that $x_n < x_{n-1}$ for all $n \in \mathbb{N}$.

c) Prove that the sequence x_n is convergent.

d) Determine the limit of the sequence x_n.

Problem 4. For $x \in \mathbb{R}$ with $x > 0$ let

$$f(x) = 2 + \frac{1}{x}.$$

a) Show that $f : [2, \infty) \to [2, \infty)$ is a contraction.

b) Define the sequence x_n by $x_0 = 1$ and

$$x_n = 2 + \frac{1}{x_{n-1}}.$$

Why is this sequence convergent? What is its limit?

Problem 5. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Explain why the maximum norm

$$|f| = \max_{0 \le x \le 1} |f(x)|$$

is well defined.

Problem 6. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Explain why f_n is not a Cauchy sequence in the maximum norm.