Vrije Universiteit Amsterdam — **March 13 2019**
Mathematical Analysis, (rehearsal) Exam 1 — **Docent: Joost Hulshof**

Write the calculations and arguments that lead to your answers.
Motivate your answers (mention theorems used).
You can use *earlier* statements, even if you failed to prove them.
NO calculators/computers/phones allowed.

For this exam you would get 2.45 hours. The real exam is only 2 hours. Your grade will be $1+\frac{\text{your total score}}{\text{total maximal score}}$.

Problem 1. Some basic theory.

a) Formulate the Archimedean Principle.

b) Give the definition of a Cauchy sequence.

c) Give the definition of a convergent sequence.

d) Formulate the Bolzano Weierstrass Theorem.

Problem 2. Consider the sequence x_n indexed by $n \in \mathbb{N}$ defined by

$$x_n = \frac{1}{\sqrt{n}}.$$

a) Prove that x_n is convergent.

b) Prove that 0 is the largest lower bound for the sequence x_n.

Problem 3. For $a \in (0, \frac{1}{4})$ define the sequence x_n by $x_0 = 1$ and

$$x_n = 1 - \frac{a}{x_{n-1}}.$$

a) Use induction to show that $x_n > \frac{1}{2}$ for all $n \in \mathbb{N}$.

b) Use induction to show that $x_n < x_{n-1}$ for all $n \in \mathbb{N}$.

c) Prove that the sequence x_n is convergent.

d) Determine the limit of the sequence x_n.

Problem 4. For $x \in \mathbb{R}$ with $x > 0$ let

$$f(x) = 2 + \frac{1}{x}.$$

a) Show that $f : [2, \infty) \to [2, \infty)$ is a contraction.

b) Define the sequence x_n by $x_0 = 1$ and

$$x_n = 2 + \frac{1}{x_{n-1}}.$$

Why is this sequence convergent? What is its limit?

Problem 5. For $x \in \mathbb{R}$ with $x > 0$ let

$$f(x) = \frac{1}{x^2} + \frac{1}{x} + x^2 \quad \text{and let} \quad m = \inf\{f(x) : x > 0\}.$$

a) Prove there exists a sequence $x_n > 0$ with $f(x_n) \to m$ as $n \to \infty$ and show that $m \leq 3$.

b) Prove that m is a global positive minimum of f. Hint: use a) and the Bolzano Weierstrass Theorem.

Problem 6. For $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{n^2}{x^2 + n^2}.$$

a) Let $x \in \mathbb{R}$. Prove that $f_n(x) \to 1$ as $n \to \infty$.

b) Is the sequence f_n uniformly convergent on $[0, 1]$? Hint $f_n(x)$ depends only on $\frac{x}{n}$.