Department of Computer Science
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Final Exam for *Logic and Sets*
April 1, 2022, 12:15–14:30

This exam has 2 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

1. **Island of liars and truth speakers** (12 points)
On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth). You meet three islanders A, B and C.
A says: "B or C is a truth speaker, but not both."
B says: "A is a truth speaker."
C says: "B is a truth speaker."

You need to determine, *by means of a truth table*, which of these three islanders speak the truth and which ones lie.

2. **CNF** (8 points)
Build a formula in CNF that corresponds to the following truth table (using the method explained in the third logic lecture).

p	q	r	?
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

3. **Logic circuit** (10 points)
Give a logic circuit, using only AND, OR and NOT gates, that represents $x \to y$.

4. **Predicate logic** (6+9 points)
Consider the two formulas $\exists x\, C(x)$ and $C(y)$.

   a) Which of the two formulas semantically entails the other? Motivate your answer.

   b) Give a model to show that the two formulas are not semantically equivalent.

5. Sets (9 points)
Consider the following set-theoretic equality

$$(A\backslash B) \cup ((B \cup C')\backslash B) \;=\; ((A' \cap C) \cup B)'.$$

Prove it using the rules of the algebra of sets.

6. Relations (3+5+4 points)
In the set $V = \{1, 2, 3, 4\}$ consider the relation R given by the set of pairs

$$\{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}.$$

a) Show that R is not an equivalence relation.

b) Show that R is an ordering relation which is not total.

c) Determine the minimal and maximal elements of R. Does there exist a largest element? Does there exist a smallest element?

7. Functions (6+6 points)
We are given the functions $root : \mathbb{R} \to \mathbb{R}$, $exp : \mathbb{R} \to \mathbb{R}$, and $add : \mathbb{R} \to \mathbb{R}$, defined by

$$root(x) \;=\; \sqrt{x}, \qquad exp(x) \;=\; e^x, \qquad \text{and} \qquad add(x) \;=\; x + 1.$$

a) For each of these functions check whether it is total and whether it is surjective.

b) Give a description (an expression for $f(x)$) for the function $f := root \circ add^{-1} \circ exp$ and specify the domain of definition D_f and the range R_f.

8. Induction and recursion (3+9 points)
Consider the sequence $(t_n)_{n\in\mathbb{N}}$ of numbers defined recursively by

$$t_1 := 0, \qquad t_{n+1} := t_n + \frac{1}{n(n+1)} + 1.$$

We claim that the following statement is true for all natural numbers n:

$$t_n = n - \frac{1}{n}.$$

a) Verify by explicit computation that the claim is true for $n = 1$, $n = 2$ and $n = 3$.

b) Prove by mathematical induction that the statement holds true for all natural numbers n.