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Resit Exam for the course *Logic and Sets*
June 7, 2019, 13:30–16:15

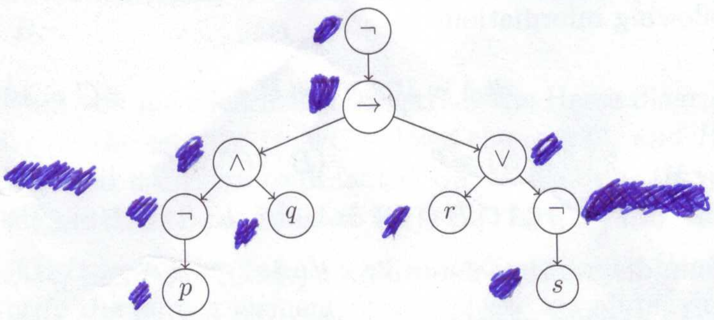
This exam has 3 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Please motivate all answers!

1. Parse tree (5+6 points)

(a) For the following parse tree, find the propositional logic formula that it represents.



(b) Draw how the truth value of this formula is determined by means of its parse tree in a bottom-up fashion, if $p = \mathsf{F}$, $q = \mathsf{F}$, $r = \mathsf{F}$, and $s = \mathsf{T}$.

2. Disjunctive normal form (3+6 points)

(a) Construct the truth table for the formula $(p \vee q) \rightarrow \neg r$.

(b) Extract from this truth table a formula in DNF that is semantically equivalent to $(p \vee q) \rightarrow \neg r$.

3. OBDDs (3+8 points)

(a) Represent $x \wedge (y \vee z)$ by means of a binary decision tree, with respect to the variable ordering x, y, z.

(b) Reduce this binary decision tree to an ordered binary decision diagram. (Also give the intermediate reduction steps.)

4. Predicate logic (7+7 points)
For each of the following pairs of formulas, either argue that they are semantically equivalent, or give a model on which their truth values are different. In cases where semantic equivalence does not hold, explain moreover whether one of the formulas semantically entails the other.

(a) $\exists x\,(C(x) \vee D(x))$ and $(\exists x\, C(x)) \vee (\exists x\, D(x))$

(b) $\exists x\,(C(x) \wedge D(x))$ and $(\exists x\, C(x)) \wedge (\exists x\, D(x))$

5. Sets (6 + 6 points)

(a) Suppose we have three sets A, B and C in a universe U, about which we have the following information:

$$\#A = 10, \qquad \#B = 7, \qquad \#C = 10,$$

$$\#(A \cap B) = 6, \qquad \#(B \cap C) = 5, \qquad \#(C \cap A) = 7,$$

$$\#(A \cap B \cap C) = 4, \qquad \#(\,(A \cup B \cup C)'\,) = 5.$$

Use Venn diagrams to determine $\#(A \cup B \cup C)$ and $\#U$.

(b) Using the algebra of sets, prove that the following set-theoretic equality holds for all sets A, B and C:

$$A \cap (B' \cup (B \cap C')) = (A' \cup (B \cap C))'.$$

6. Relations and functions (3 + 4 + 4 points)
In the set $V := \{1, 2, 3, 4, 5\}$, consider the binary relations R and S defined by

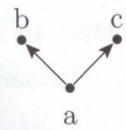
$$R := \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 4,1\rangle, \langle 5,4\rangle\},$$
$$S := \{\langle 2,1\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle, \langle 4,5\rangle\}.$$

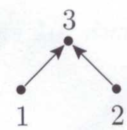
(a) Draw the directed graph representations of R and S.

(b) For each of the two relations R and S, decide whether or not it is a function (from V to V). Please explain your answers.

(c) Determine the relation $S^{-1} \circ R$, and give an explicit listing of its elements.

7. **Ordering relations** (8 + 3 points)
Let R and S be the partial ordering relations on the respective sets $A := \{\texttt{a}, \texttt{b}, \texttt{c}\}$ and $B := \{1, 2, 3\}$, specified by the following Hasse diagrams:

R: 

S: 

Consider the product set

$$A \times B = \{\langle \texttt{a}, 1\rangle, \langle \texttt{a}, 2\rangle, \langle \texttt{a}, 3\rangle, \langle \texttt{b}, 1\rangle, \langle \texttt{b}, 2\rangle, \langle \texttt{b}, 3\rangle, \langle \texttt{c}, 1\rangle, \langle \texttt{c}, 2\rangle, \langle \texttt{c}, 3\rangle\}.$$

We are interested in the lexicographic order on $A \times B$ induced by the partial orders R on A, and S on B.

(a) Use the algorithm you have learned to construct the Hasse diagram of the lexicographic order on the set $A \times B$: write down the sets G_x and H_x obtained in the construction, and use them to draw the Hasse diagram. (If you prefer, you can simplify the notation by writing a1 for $\langle \texttt{a}, 1\rangle$, a2 for $\langle \texttt{a}, 2\rangle$, and so on.)

(b) Does the set $A \times B$ have a largest element according to your Hasse diagram? If so, please identify the largest element; if not, please list all maximal elements.

8. **Induction** (3 + 8 points)
Consider the sequence $(t_n)_{n=1}^{\infty}$ of integer numbers defined recursively by

$$t_1 := 0, \qquad t_{n+1} = t_n + n(3n + 1).$$

We claim that for all $n \geq 1$,

$$t_n = n^2(n - 1).$$

(a) Verify by explicit calculation that the claim is true for $n = 1$, $n = 2$, and $n = 3$.

(b) Prove by mathematical induction that the claim is true for *all* integers $n \geq 1$.