Department of Computer Science
Vrije Universiteit Amsterdam

Midterm Exam for *Logic and Sets*
February 28, 2019, 9:30–11:30

This exam has 3 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Please motivate all answers!

1. Semantic entailment (5 + 5 points)

(a) Suppose the semantic entailment $\phi_1 \models \phi_2$ holds. Can we conclude that then the semantic entailment $(\phi_1 \to \psi) \models (\phi_2 \to \psi)$ holds? If so, show this by means of a logical argumentation or a truth table; if not, give a counterexample.

(b) Suppose the semantic entailment $\psi_1 \models \psi_2$ holds. Can we conclude that then the semantic entailment $(\phi \to \psi_1) \models (\phi \to \psi_2)$ holds? If so, show this by means of a logical argumentation or a truth table; if not, give a counterexample.

2. Island puzzle (8 points)
On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth). Suppose an islander A would say: "If I am a liar, then I am a truth speaker."

Determine via a truth table whether A would be a truth speaker, a liar, both options are possible, or this statement is impossible on the island.

3. Conjunctive normal form (8 points)
Give the truth table of the DNF $(p \wedge q) \vee r$, and use it to construct a formula in CNF that is semantically equivalent.

4. `CNF` algorithm (10 points)
Apply the algorithm `CNF` to turn the formula $\neg(p \to q) \vee (\neg p \wedge q)$ into CNF.

5. DPLL procedure (9 points)
Apply the DPLL procedure to the CNF $(\neg p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee \neg q)$, to check whether it is satisfiable.

6. Sets (6 + 9 points)
In this task, we consider the following set-theoretic equality:

$$(A' \cup (A \cap B'))' = A \cap B$$

(a) Draw Venn diagrams for the two sets $(A' \cup (A \cap B'))'$ and $A \cap B$, depicting clearly which area corresponds to the set, and how the set is constructed from A and B using the fundamental set operations. Use your Venn-diagrams to conclude that the set-theoretic equality displayed above holds.

(b) Prove the set-theoretic equality displayed above (for all sets A and B) using the rules of the algebra of sets.

7. Relations (4 + 6 + 6 points)
Consider the relations R and S, both in the set $V := \{1, 2, 3, 4\}$, given by the following matrix representations:

$$R\colon \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad S\colon \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(rows and columns are numbered in the order $1, 2, 3, 4$)

(a) Is the relation S reflexive? transitive? symmetric? anti-symmetric? Briefly explain your answers.

(b) Explicitly list all the elements of $R \circ S$ in the curly-bracket notation and draw the directed graph representation of $R \circ S$.

(c) Find a relation in the set V that is *neither* symmetric *nor* anti-symmetric, and draw its directed graph representation.

8. Ordering relations (8 + 3 + 3 points)
In this task, we consider the Cartesian ordering relation in the set

$$A := \{\mathtt{a}, \mathtt{b}\}^3 = \{\mathtt{aaa}, \mathtt{aab}, \mathtt{aba}, \mathtt{abb}, \mathtt{baa}, \mathtt{bab}, \mathtt{bba}, \mathtt{bbb}\},$$

induced by the alphabetic order on $\{\mathtt{a}, \mathtt{b}\}$. Note that we write $\mathtt{aba}$ instead of $\langle \mathtt{a}, \mathtt{b}, \mathtt{a} \rangle$, and so on, to simplify the notation.

(a) Use the algorithm that you have learned to construct the Hasse diagram that represents the Cartesian ordering relation on A. Explicitly write down the sets G_x and H_x obtained in the construction, then draw the Hasse diagram.

Let B be the subset of A that is given by

$$B := \{\texttt{aaa}, \texttt{aab}, \texttt{aba}, \texttt{abb}, \texttt{bab}\}.$$

(b) Does the set B have a smallest element according to the Cartesian ordering relation from part (a)? If so, please give this smallest element. If not, please list all the minimal elements of the set B.

(c) Does the set B have a largest element according to the Cartesian ordering relation from part (a)? If so, please give this largest element. If not, please list all the maximal elements of the set B.