This exam has 6 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Please motivate all answers!

1. Semantic entailment (5 + 5 points)

(a) Suppose the semantic entailment $\phi_1 \models \phi_2$ holds. Can we conclude that then the semantic entailment $(\phi_1 \to \psi) \models (\phi_2 \to \psi)$ holds? If so, show this by means of a logical argumentation or a truth table; if not, give a counterexample.

Solution: The semantic entailment does not hold in general. Take ϕ_1 to be F, ϕ_2 to be T, and ψ_2 to be F. Then $\phi_1 \models \phi_2$. But, since $\phi_1 \to \psi$ is T and $\phi_2 \to \psi$ is F, $(\phi_1 \to \psi) \models (\phi_2 \to \psi)$ does not.

(b) Suppose the semantic entailment $\psi_1 \models \psi_2$ holds. Can we conclude that then the semantic entailment $(\phi \to \psi_1) \models (\phi \to \psi_2)$ holds? If so, show this by means of a logical argumentation or a truth table; if not, give a counterexample.

Solution: We can conclude that the semantic entailment holds. The logical argumentation is that $\psi_1 \models \psi_2$ implies that $\psi_1 \to \psi_2$ is a tautology. So if $\phi \to \psi_1$ is T, then (by transitivity of implication) $\phi \to \psi_2$ is T.

The argumentation via a truth table is:

ϕ	ψ_1	ψ_2	$\phi \to \psi_1$	$\phi \to \psi_2$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Since $\psi_1 \models \psi_2$, rows 2 and 6 where ψ_1 is T and ψ_2 is F are ruled out. In the remaining rows, $\phi \to \psi_1$ is F or $\phi \to \psi_2$ is T.

2. Island puzzle (8 points)

On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth). Suppose an islander A would say: “If I am a liar, then I am a truth speaker.”

Determine via a truth table whether A would be a truth speaker, a liar, both options are possible, or this statement is impossible on the island.

Solution: The formalization of the statement is $t_A \leftrightarrow (\neg t_A \rightarrow t_A)$.

t_A	$\neg t_A$	$\neg t_A \rightarrow t_A$	$t_A \leftrightarrow (\neg t_A \rightarrow t_A)$
T	F	T	T
F	T	F	T

Since the fourth column contains two `T` values, both options are possible: A can be a truth speaker or a liar.

3. Conjunctive normal form (8 points)
Give the truth table of the DNF $(p \wedge q) \vee r$, and use it to construct a formula in CNF that is semantically equivalent.

Solution:

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

Since only the fourth, sixth and eighth row of the fifth column contain `F`, the resulting CNF is

$$(\neg p \vee q \vee r) \;\wedge\; (p \vee \neg q \vee r) \;\wedge\; (p \vee q \vee r)$$

4. `CNF` algorithm (10 points)
Apply the algorithm `CNF` to turn the formula $\neg(p \rightarrow q) \vee (\neg p \wedge q)$ into CNF.

Solution:

$$\begin{aligned}
\neg(p \rightarrow q) \vee (\neg p \wedge q) \quad & \rightsquigarrow_{\texttt{IMPL-FREE}} \neg(\neg p \vee q) \vee (\neg p \wedge q) \\
& \rightsquigarrow_{\texttt{NNF}} (\neg\neg p \wedge \neg q) \vee (\neg p \wedge q) \\
& \rightsquigarrow_{\texttt{NNF}} (p \wedge \neg q) \vee (\neg p \wedge q) \\
& \rightsquigarrow_{\texttt{DISTR}} ((p \wedge \neg q) \vee \neg p) \wedge ((p \wedge \neg q) \vee q) \\
& \rightsquigarrow_{\texttt{DISTR}} (p \vee \neg p) \wedge (\neg q \vee \neg p) \wedge ((p \wedge \neg q) \vee q) \\
& \rightsquigarrow_{\texttt{DISTR}} (p \vee \neg p) \wedge (\neg q \vee \neg p) \wedge (p \vee q) \wedge (\neg q \vee q)
\end{aligned}$$

5. DPLL procedure (9 points)
Apply the DPLL procedure to the CNF $(\neg p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee \neg q)$, to check whether it is satisfiable.

Solution: First try p is $\top$:

$$(\neg\top \vee \neg q) \wedge (\neg\top \vee q) \wedge (\top \vee \neg q)$$

This reduces to $q \wedge \neg q$. Both attempting $\top$ or $\bot$ for q yields $\bot$, so the temporary verdict is "unsatisfiable".

Now try p is $\bot$:

$$(\neg\bot \vee \neg q) \wedge (\neg\bot \vee q) \wedge (\bot \vee \neg q)$$

This reduces to $\neg q$. Attempting $\top$ for q yields $\bot$, giving the temporary verdict "unsatisfiable". Finally, attempting $\bot$ for q yields $\top$. So the final verdict is "satisfiable", if both p and q have the value $\bot$.

6. Sets (6 + 9 points)
In this task, we consider the following set-theoretic equality:

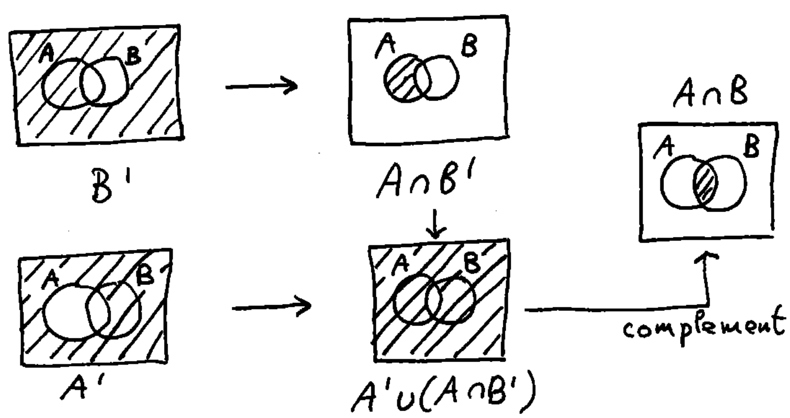
$$(A' \cup (A \cap B'))' = A \cap B$$

(a) Draw Venn diagrams for the two sets $(A' \cup (A \cap B'))'$ and $A \cap B$, depicting clearly which area corresponds to the set, and how the set is constructed from A and B using the fundamental set operations. Use your Venn-diagrams to conclude that the set-theoretic equality displayed above holds.

(b) Prove the set-theoretic equality displayed above (for all sets A and B) using the rules of the algebra of sets.

Solution:

(a) The following Venn-diagrams confirm that the equality holds:



(b) $$\begin{aligned} (A' \cup (A \cap B'))' &= (A')' \cap (A \cap B')' && \text{(De Morgan)} \\ &= A \cap (A \cap B')' && \text{(involution)} \\ &= A \cap (A' \cup (B')') && \text{(De Morgan)} \\ &= A \cap (A' \cup B) && \text{(involution)} \\ &= (A \cap A') \cup (A \cap B) && \text{(distributivity)} \\ &= \varnothing \cup (A \cap B) && \text{(complement)} \\ &= A \cap B && \text{(identity).} \end{aligned}$$

7. Relations (4 + 6 + 6 points)
Consider the relations R and S, both in the set $V := \{1, 2, 3, 4\}$, given by the following matrix representations:

$$R\colon \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad\qquad S\colon \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(rows and columns are numbered in the order $1, 2, 3, 4$)

(a) Is the relation S reflexive? transitive? symmetric? anti-symmetric? Briefly explain your answers.

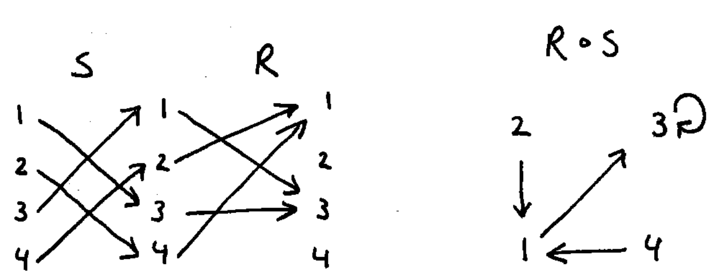
(b) Explicitly list all the elements of $R \circ S$ in the curly-bracket notation and draw the directed graph representation of $R \circ S$.

(c) Find a relation in the set V that is *neither* symmetric *nor* anti-symmetric, and draw its directed graph representation.

Solution:

(a) Reflexive: no, no element relates to itself.
Transitive: no, we have $1\,S\,3$ and $3\,S\,1$, but not $1\,S\,1$.
Symmetric: yes, whenever an element x relates to y, y also relates to x.
Anti-symmetric: no, we have $1\,S\,3$ but also $3\,S\,1$.

(b) We draw a Venn representation of the composition $R \circ S$:

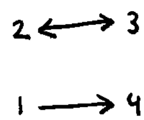


From the Venn representation we can read off all the pairs that together make up the composition:

$$R \circ S = \{\langle 1, 3\rangle, \langle 2, 1\rangle, \langle 3, 3\rangle, \langle 4, 1\rangle\}$$

We can now draw the directed graph representation; again see the picture above.

(c) An example is the relation depicted in the following directed graph:



8. Ordering relations (8 + 3 + 3 points)
In this task, we consider the Cartesian ordering relation in the set

$$A := \{\mathtt{a}, \mathtt{b}\}^3 = \{\mathtt{aaa}, \mathtt{aab}, \mathtt{aba}, \mathtt{abb}, \mathtt{baa}, \mathtt{bab}, \mathtt{bba}, \mathtt{bbb}\},$$

induced by the alphabetic order on $\{\mathtt{a}, \mathtt{b}\}$. Note that we write $\mathtt{aba}$ instead of $\langle \mathtt{a}, \mathtt{b}, \mathtt{a}\rangle$, and so on, to simplify the notation.

(a) Use the algorithm that you have learned to construct the Hasse diagram that represents the Cartesian ordering relation on A. Explicitly write down the sets G_x and H_x obtained in the construction, then draw the Hasse diagram.

Let B be the subset of A that is given by

$$B := \{\mathtt{aaa}, \mathtt{aab}, \mathtt{aba}, \mathtt{abb}, \mathtt{bab}\}.$$

(b) Does the set B have a smallest element according to the Cartesian ordering relation from part (a)? If so, please give this smallest element. If not, please list all the minimal elements of the set B.

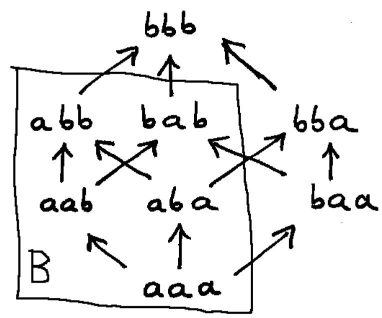
(c) Does the set B have a largest element according to the Cartesian ordering relation from part (a)? If so, please give this largest element. If not, please list all the maximal elements of the set B.

Solution:

(a) The algorithm gives the following sets G_x and H_x:

$$\begin{aligned}
G_{\texttt{aaa}} &= \{\texttt{aab}, \texttt{aba}, \texttt{abb}, \texttt{baa}, \texttt{bab}, \texttt{bba}, \texttt{bbb}\} & H_{\texttt{aaa}} &= \{\texttt{aab}, \texttt{aba}, \texttt{baa}\} \\
G_{\texttt{aab}} &= \{\texttt{abb}, \texttt{bab}, \texttt{bbb}\} & H_{\texttt{aab}} &= \{\texttt{abb}, \texttt{bab}\} \\
G_{\texttt{aba}} &= \{\texttt{abb}, \texttt{bba}, \texttt{bbb}\} & H_{\texttt{aba}} &= \{\texttt{abb}, \texttt{bba}\} \\
G_{\texttt{baa}} &= \{\texttt{bab}, \texttt{bba}, \texttt{bbb}\} & H_{\texttt{baa}} &= \{\texttt{bab}, \texttt{bba}\} \\
G_{\texttt{abb}} &= \{\texttt{bbb}\} & H_{\texttt{abb}} &= \{\texttt{bbb}\} \\
G_{\texttt{bab}} &= \{\texttt{bbb}\} & H_{\texttt{bab}} &= \{\texttt{bbb}\} \\
G_{\texttt{bba}} &= \{\texttt{bbb}\} & H_{\texttt{bba}} &= \{\texttt{bbb}\} \\
G_{\texttt{bbb}} &= \varnothing & H_{\texttt{bbb}} &= \varnothing
\end{aligned}$$

The Hasse diagram therefore looks like this:



(b) As the Hasse diagram shows, the smallest element in the set B is `aaa`.

(c) As the Hasse diagram shows, the set B does not have a largest element. It has two maximal elements: `abb` and `bab`.