This exam has 3 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. Semantic entailment / CNF (6+5 points)

(a) Decide by means of a truth table whether

$$(q \to p) \land \neg p \models \neg q$$

is a valid semantic entailment.

(b) Use the truth table to construct a CNF that is semantically equivalent to

$$(q \to p) \land \neg p$$

2. Island puzzle (10 points)
On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth).

You meet three islanders A, B and C.
A says: "B or C is a liar."
B says: "C is a liar."
C says: "if A is a liar, then B is a liar."

Determine, either through a truth table or by means of logical reasoning using propositional formulas, which of these three islanders speak the truth and which ones lie.

Also check explicitly that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars.

3. OBDDs (3+8 points)

(a) Represent $x \oplus y \oplus z$ by means of a binary decision tree, with respect to the variable ordering x, y, z.

(b) Reduce this binary decision tree to an ordered binary decision diagram. (Also give the intermediate reduction steps.)

4. Predicate logic (3 + 3 + 7 points)
Consider the following two unary predicates:

- $F(x)$: x is female
- $D(x)$: x wears a dress

(a) Express the phrase "Everybody who wears a dress, is female" in predicate logic.

(b) Express the phrase "If everybody wears a dress, then everybody is female" in predicate logic.

(c) Are the formulas in (a) and (b) semantically equivalent? If so, argue they coincide on all models. If not, give a model on which they produce different truth values.

5. Sets (5+6 points)

(a) In a universe U of 40 elements, we have three sets A, B and C, about which we know the following:

$$\#A = 18 \qquad \#B = 15 \qquad \#C = 16$$
$$\#(A \cap B) = 7 \qquad \#(B \cap C) = 7 \qquad \#(A \cap C) = 9$$
$$\#(A \cap B \cap C) = 4$$

Use a Venn diagram to determine the numbers

$$\#(A \cup B \cup C)' \qquad \text{and} \qquad \#((A \cup B) \setminus C').$$

(b) Prove the following identity (for all sets A, B and C) using the algebra of sets:

$$(B \cap A)' \setminus (B \cap C') = B' \cup (A' \cap C).$$

NB: the laws of the algebra of sets are on the last page of this exam.

6. Ordering relations (8+4 points)
Consider the set

$$A := \{2, 3, 5, 6, 10, 15, 20, 30\}$$

together with the partial ordering relation *IsDivisorOf* in this set (i.e., $\langle x, y \rangle \in$ *IsDivisorOf* if and only if x is a divisor of y).

(a) Apply the algorithm you have learned to construct the Hasse diagram of the partial ordering relation *IsDivisorOf* in the set A. Explicitly write down the sets G_x and H_x obtained in the construction, and draw the Hasse diagram.

(b) Does the set A have a largest element according to the *IsDivisorOf* relation? If so, please give this largest element. If not, please list all the maximal elements of the set A.

7. Relations and functions (6+6 points)

In the set $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$, we consider the three relations Q, R and S defined explicitly as follows:

$$Q := \{\langle \mathtt{a}, \mathtt{c}\rangle, \langle \mathtt{b}, \mathtt{a}\rangle, \langle \mathtt{c}, \mathtt{b}\rangle, \langle \mathtt{d}, \mathtt{d}\rangle\}$$
$$R := \{\langle \mathtt{a}, \mathtt{c}\rangle, \langle \mathtt{b}, \mathtt{a}\rangle, \langle \mathtt{c}, \mathtt{d}\rangle, \langle \mathtt{d}, \mathtt{c}\rangle\}$$
$$S := \{\langle \mathtt{b}, \mathtt{d}\rangle, \langle \mathtt{c}, \mathtt{a}\rangle, \langle \mathtt{c}, \mathtt{c}\rangle, \langle \mathtt{d}, \mathtt{d}\rangle\}$$

(a) For each of the three relations Q, R and S, determine whether or not it is an *injective function*. In each case, explain your answer.

(b) Use a Venn diagram to determine the composite relation $Q \circ R^{-1} \circ S$. List all the elements of $Q \circ R^{-1} \circ S$ explicitly using the curly-bracket notation.

8. Induction (10 points)

Prove, using mathematical induction, that for all $n \in \mathrm{N}$,

$$\sum_{k=0}^{n}(2k-1) = (n+1)(n-1).$$

NB: recall that $\sum_{k=0}^{n}(2k-1)$ stands for $(2\cdot 0-1)+(2\cdot 1-1)+(2\cdot 2-1)+\cdots+(2\cdot n-1)$.