Department of Computer Science
Vrije Universiteit Amsterdam

Final Exam for the course *Logic and Sets*
March 29, 2018, 12:00–14:00

This exam has 3 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. Equivalence relations (7 + 8 points)
On the set $A := \{1,2,3,4\} \times \{1,2,3,4\}$, consider the relation R that is defined by the description

$$\langle n_1, k_1 \rangle \; R \; \langle n_2, k_2 \rangle \qquad \iff \qquad |n_1 - k_1| = |n_2 - k_2|$$

(a) Show that R is an equivalence relation.

(b) Explicitly write down all the different equivalence classes of the equivalence relation R on A, and give a complete system of representatives.

2. Injections and Surjections (4 + 4 points)
For each of the two functions defined below, determine whether or not the function is injective, and whether or not the function is surjective, and explain your answers.

(a) The function $f \colon \mathrm{Z} \to \mathrm{N}$ defined by $f(n) = |n - 2|$.

(b) The function $g \colon \mathit{CSstudents} \to \mathrm{N}$ that assigns to every Computer Science student at VU Amsterdam his or her student number.

3. Function composition (4 + 4 points)
We are given the functions $\mathit{exp}_2 \colon \mathrm{R} \to \mathrm{R}$, $\mathit{inv} \colon \mathrm{R} \to \mathrm{R}$ and $\mathit{inc} \colon \mathrm{R} \to \mathrm{R}$ defined by

$$\mathit{exp}_2(x) = 2^x \qquad \mathit{inv}(x) = \frac{1}{x} \qquad \mathit{inc}(x) = x + 1$$

(a) Give an explicit description (i.e. a formula for $f(x)$) of the function f defined by

$$f := \mathit{exp}_2 \circ \mathit{inv} \circ \mathit{inc}.$$

(b) Express the function $g \colon \mathrm{R} \to \mathrm{R}$ defined by

$$g(x) := \frac{1}{2^{x-2}}$$

as a composition of the functions exp_2, inv and inc and their inverses.

4. Induction (4 + 10 points)
Consider the sequence $(t_n)_{n=1}^{\infty}$ of numbers defined recursively by

$$t_1 := 0, \qquad t_{n+1} := t_n + 2(n+1).$$

(a) Calculate the terms t_2, t_3, t_4 and t_5 of this sequence.

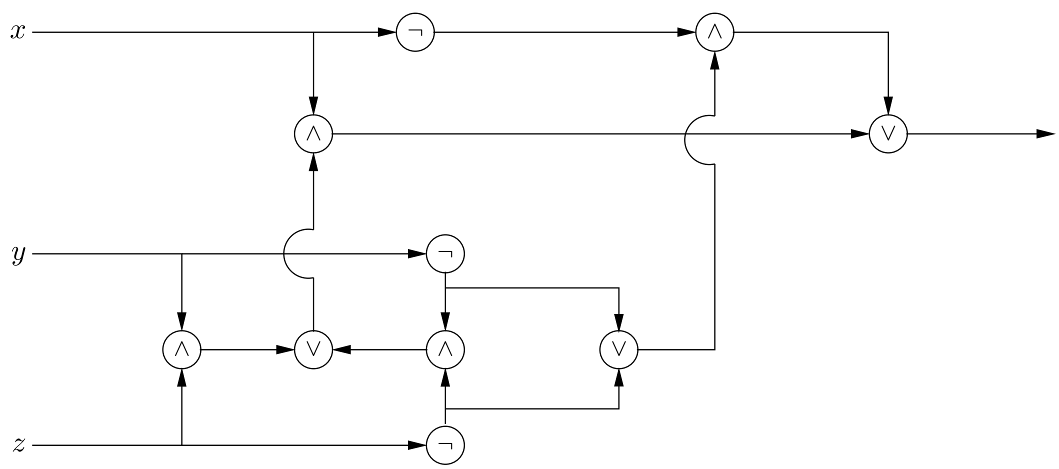
(b) Prove by mathematical induction that for all $n \geq 1$,

$$t_n = (n+2)(n-1).$$

5. Axioms for semantic equivalence (12 points)
Show using the axioms for semantic equivalence that

$$\phi \land (\psi \lor (\neg\phi \land \chi)) \;\equiv\; \phi \land \psi$$

6. Logic circuit (8 points)
Give the propositional formula that corresponds to the following logic circuit:



7. OBDD (4 + 8 points)

(a) Represent $(x \oplus y) \land z$ by means of a binary decision tree, with respect to the variable ordering $x,\ y,\ z$.

(b) Reduce this binary decision tree to an ordered binary decision diagram.

8. **Predicate logic** (3 + 3 + 7 points)
Suppose there is a collection of dolls and there are two boxes. Consider the following three unary predicates:

- $D(x)$: x is a doll
- $B_1(x)$: x is in the first box
- $B_2(x)$: x is in the second box

(a) Express the phrase "Every doll is in one of the two boxes" in predicate logic.

(b) Express the phrase "All dolls are in the first box or all dolls are in the second box" in predicate logic.

(c) Are the formulas in (a) and (b) semantically equivalent? If so, argue they coincide on all models. If not, give a model on which they produce different truth values.