This exam has 6 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. Equivalence relations (7 + 8 points)
On the set $A := \{1,2,3,4\} \times \{1,2,3,4\}$, consider the relation R that is defined by the description

$$\langle n_1, k_1 \rangle \; R \; \langle n_2, k_2 \rangle \qquad \Longleftrightarrow \qquad |n_1 - k_1| = |n_2 - k_2|$$

(a) Show that R is an equivalence relation.

(b) Explicitly write down all the different equivalence classes of the equivalence relation R on A, and give a complete system of representatives.

Solution:

(a) We must show that R is reflexive, symmetric, and transitive.

(i) Since $|n-k| = |n-k|$, $\langle n, k\rangle \, R \, \langle n, k \rangle$ for all $\langle n, k \rangle \in A$. Hence, R is reflexive.

(ii) Suppose $\langle n_1, k_1 \rangle \; R \; \langle n_2, k_2 \rangle$. Then $|n_1 - k_1| = |n_2 - k_2|$, hence $|n_2 - k_2| = |n_1 - k_1|$, and therefore $\langle n_2, k_2 \rangle \; R \; \langle n_1, k_1 \rangle$. So R is symmetric.

(iii) Suppose $\langle n_1, k_1 \rangle \, R \, \langle n_2, k_2 \rangle$ and $\langle n_2, k_2 \rangle \, R \, \langle n_3, k_3 \rangle$. Then $|n_1 - k_1| = |n_2 - k_2|$ and $|n_2 - k_2| = |n_3 - k_3|$, which implies that $|n_1 - k_1| = |n_3 - k_3|$, and therefore, $\langle n_1, k_1 \rangle \; R \; \langle n_3, k_3 \rangle$. So R is transitive.

(b) The equivalence classes are

$$\begin{aligned}
&\{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\} \\
&\{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,4\rangle, \langle 4,3\rangle\} \\
&\{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle\} \\
&\{\langle 1,4\rangle, \langle 4,1\rangle\}
\end{aligned}$$

Therefore, a complete system of representatives is

$$\{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle\}$$

2. Injections and Surjections (4 + 4 points)
For each of the two functions defined below, determine whether or not the function is injective, and whether or not the function is surjective, and explain your answers.

(a) The function $f\colon \mathrm{Z} \to \mathrm{N}$ defined by $f(n) = |n-2|$.

(b) The function $g\colon \mathit{CSstudents} \to \mathrm{N}$ that assigns to every Computer Science student at VU Amsterdam his or her student number.

Solution:

(a) f is not injective because, for example, $f(1) = |-1| = f(3) = |1| = 1$. f is surjective because for every $k \in \mathrm{N}$, $f(k+2) = |k| = k$.

(b) Since different Computer Science students have different student numbers, g is injective. g is not surjective, because the number of Computer Science students is finite, while the codomain of g is infinite.

3. Function composition (4 + 4 points)
We are given the functions $\mathit{exp}_2\colon \mathrm{R} \to \mathrm{R}$, $\mathit{inv}\colon \mathrm{R} \to \mathrm{R}$ and $\mathit{inc}\colon \mathrm{R} \to \mathrm{R}$ defined by

$$\mathit{exp}_2(x) = 2^x \qquad \mathit{inv}(x) = \frac{1}{x} \qquad \mathit{inc}(x) = x + 1$$

(a) Give an explicit description (i.e. a formula for $f(x)$) of the function f defined by

$$f := \mathit{exp}_2 \circ \mathit{inv} \circ \mathit{inc}.$$

(b) Express the function $g\colon \mathrm{R} \to \mathrm{R}$ defined by

$$g(x) := \frac{1}{2^{x-2}}$$

as a composition of the functions exp_2, inv and inc and their inverses.

Solution:

(a) $f(x) = 2^{\frac{1}{x+1}}$

(b) $g = \mathit{inv} \circ \mathit{exp}_2 \circ \mathit{inc}^{-1} \circ \mathit{inc}^{-1}$

4. Induction (4 + 10 points)
Consider the sequence $(t_n)_{n=1}^{\infty}$ of numbers defined recursively by

$$t_1 := 0, \qquad t_{n+1} := t_n + 2(n+1).$$

(a) Calculate the terms t_2, t_3, t_4 and t_5 of this sequence.

(b) Prove by mathematical induction that for all $n \geq 1$,

$$t_n = (n+2)(n-1).$$

Solution:

(a) $t_2 = t_1 + 4 = 4$, $t_3 = t_2 + 6 = 10$, $t_4 = t_3 + 8 = 18$, $t_5 = t_4 + 10 = 28$.

(b) First, $t_1 = 0$ by definition, but also $(1+2)(1-1) = 3 \cdot 0 = 0$. Therefore, $t_n = (n+2)(n-1)$ holds for $n = 1$. This proves the base case of the induction.

Now for the inductive step. Assume that $t_m = (m+2)(m-1)$, where $m \geq 1$ is arbitrary (this is our inductive hypothesis). By the recursive definition and the inductive hypothesis,

$$\begin{aligned} t_{m+1} &= t_m + 2(m+1) && \text{(by definition)} \\ &= (m+2)(m-1) + 2(m+1) && \text{(by the inductive hypothesis)} \\ &= m^2 + 2m - m - 2 + 2m + 2 \\ &= m^2 + 3m \end{aligned}$$

On the other hand,

$$(m+1+2)(m+1-1) = (m+3)m = m^2 + 3m$$

This shows that if the inductive hypothesis is true, then also

$$t_{m+1} = (m+1+2)(m+1-1)$$

is true. This completes the inductive proof.

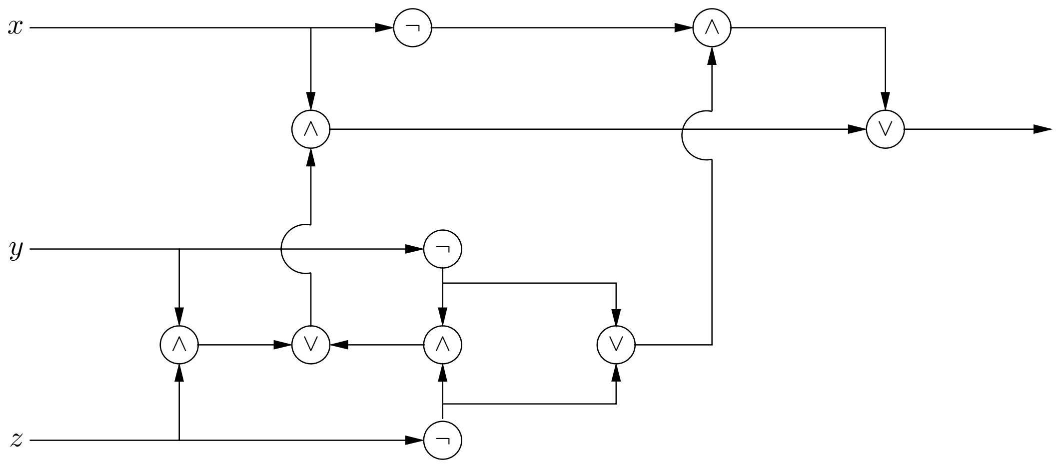
5. Axioms for semantic equivalence (12 points)
Show using the axioms for semantic equivalence that

$$\phi \wedge (\psi \vee (\neg\phi \wedge \chi)) \;\equiv\; \phi \wedge \psi$$

Solution:

$$\begin{aligned} \phi \wedge (\psi \vee (\neg\phi \wedge \chi)) &\equiv (\phi \wedge \psi) \vee (\phi \wedge (\neg\phi \wedge \chi)) && \text{(distributivity)} \\ &\equiv (\phi \wedge \psi) \vee ((\phi \wedge \neg\phi) \wedge \chi) && \text{(associativity)} \\ &\equiv (\phi \wedge \psi) \vee (\bot \wedge \chi) && \text{(complement)} \\ &\equiv (\phi \wedge \psi) \vee (\chi \wedge \bot) && \text{(commutativity)} \\ &\equiv (\phi \wedge \psi) \vee \bot && \text{(domination)} \\ &\equiv \phi \wedge \psi && \text{(identity)} \end{aligned}$$

6. **Logic circuit** (8 points)
Give the propositional formula that corresponds to the following logic circuit:



Solution: $(x \wedge ((y \wedge z) \vee (\neg y \wedge \neg z))) \vee (\neg x \wedge (\neg y \vee \neg z))$

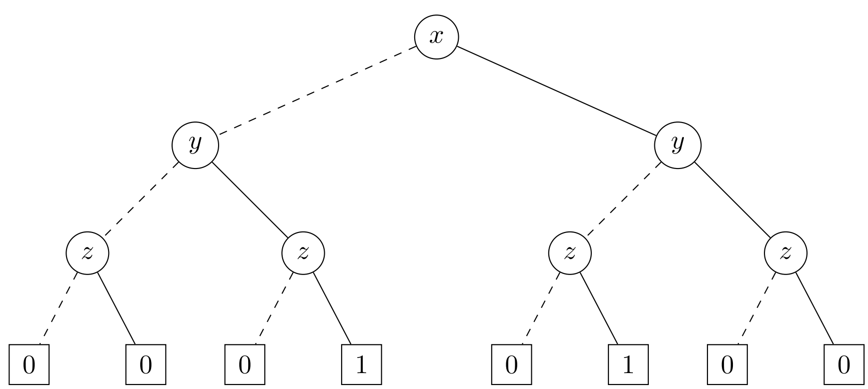
7. **OBDD** (4 + 8 points)

(a) Represent $(x \oplus y) \wedge z$ by means of a binary decision tree, with respect to the variable ordering $x,\ y,\ z$.

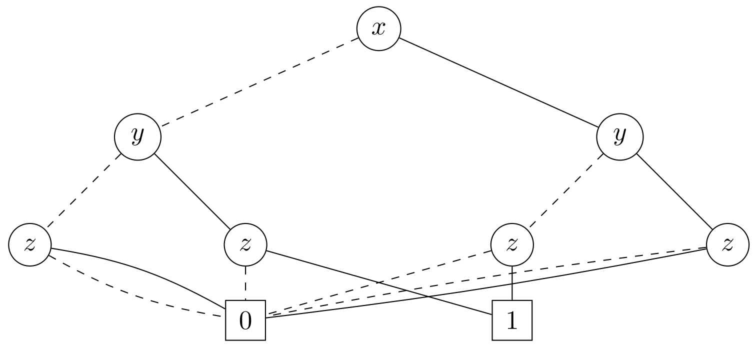
(b) Reduce this binary decision tree to an ordered binary decision diagram.

Solution:

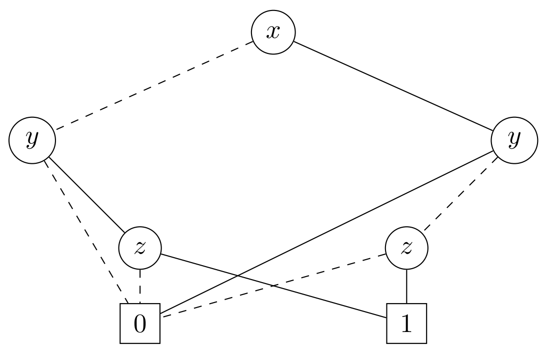
(a)



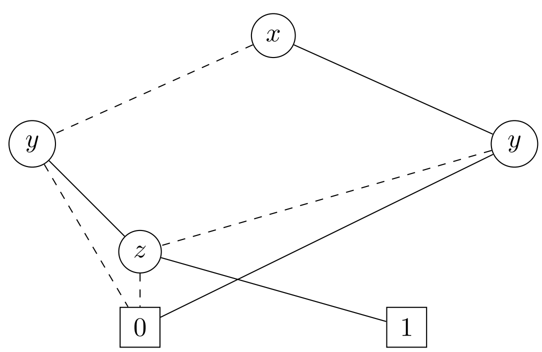
(b) First apply C1 to collapse leaves:



Next apply C2 twice to remove the most left-hand and the most right-hand z nodes:



Finally apply C3 to collapse the two remaining z nodes:



8. Predicate logic (3 + 3 + 7 points)
Suppose there is a collection of dolls and there are two boxes. Consider the following three unary predicates:

- $D(x)$: x is a doll
- $B_1(x)$: x is in the first box
- $B_2(x)$: x is in the second box

(a) Express the phrase "Every doll is in one of the two boxes" in predicate logic.

(b) Express the phrase "All dolls are in the first box or all dolls are in the second box" in predicate logic.

(c) Are the formulas in (a) and (b) semantically equivalent? If so, argue they coincide on all models. If not, give a model on which they produce different truth values.

Solution:

(a) This sentence can either be interpreted as an inclusive or

$$\forall x\,(D(x) \to B_1(x) \lor B_2(x))$$

or as an exclusive or

$$\forall x\,(D(x) \to B_1(x) \oplus B_2(x)).$$

(b) Again this sentence can either be interpreted as an inclusive or

$$\forall x\,(D(x) \to B_1(x)) \;\lor\; \forall x\,(D(x) \to B_2(x))$$

or as an exclusive or

$$\forall x\,(D(x) \to B_1(x)) \;\oplus\; \forall x\,(D(x) \to B_2(x)).$$

(c) The two formulas are not semantically equivalent.

Consider a model with a set of two elements a and b, where $D(a)$, $D(b)$, $B_1(a)$, and $B_2(b)$. (But $B_1(b)$ and $B_2(a)$ don't hold.)

The formula of part (a) holds in this model, while the formula of part (b) does not.