Department of Computer Science
Vrije Universiteit Amsterdam

Midterm Exam for the course *Logic and Sets*
February 27, 2018, 9:00–11:00

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*This exam has 4 pages and 8 exercises.*

*The result will be computed as (total number of points plus 10) divided by 10.*

*Answers may be given in either English or Dutch.*

***Please motivate all answers!***

1. **Semantic entailment** (5 + 5 points)
   Argue for each of the following two semantic entailments whether it holds. If so, show this by means of a truth table; if not, give a concrete counterexample.

   (a) $(\phi \oplus \psi) \oplus \chi \models \phi \oplus (\psi \oplus \chi)$

   (b) $\phi \to (\psi \to \chi) \models (\phi \to \psi) \to \chi$

2. **Island puzzle** (10 points)
   On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth).

   You meet three islanders $A$, $B$ and $C$.
   $A$ says: "if $B$ is a truth speaker, then $C$ is a truth speaker."
   $B$ says: "$A$ is a truth speaker or $C$ is a liar." (He means an inclusive or $\vee$.)
   $C$ says: "$A$ and $B$ are both liars."

   You need to determine, either via a truth table or by means of logical reasoning, which of these three islanders speak the truth and which ones lie.

   Also check explicitly that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars.

3. **Disjunctive normal form** (7 points)
   Give the truth table of $(p \vee q) \wedge r$, and use it to construct a formula in DNF that is semantically equivalent.

4. **Conjunctive normal form** (8 points)
   Apply the algorithm `CNF` to turn the formula $\neg(p \to q) \vee \neg(\neg p \to \neg q)$ into CNF.

5. **DPLL procedure** (10 points)
   Apply the DPLL procedure to the CNF $(p \vee q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$, to check whether it is satisfiable.

**6. Sets** (7 + 8 points)

(a) Suppose that in a universe $U$ with 40 elements, we have three sets $A$, $B$ and $C$ of which we know that

$$\#A = 15 \qquad \#B = 15 \qquad \#C = 15$$
$$\#(A \cap B) = 6 \qquad \#(A \cap C) = 8 \qquad \#(A \cap B \cap C) = 5$$
$$\#(B \setminus C) = 9.$$

Draw a Venn diagram of the sets $A$, $B$ and $C$, and determine the number of elements in every region of your Venn diagram. Then use your Venn diagram to determine the numbers

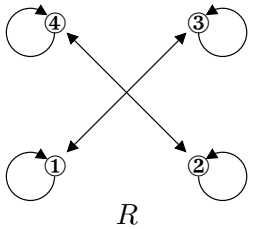
$$\#(A \setminus C) \qquad \text{and} \qquad \#(B \cup C)$$

(b) Derive the following equality (for all sets $A$, $B$ and $C$), using the algebra of sets:

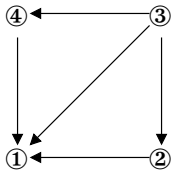
$$A \setminus (C \cap B) = (A \setminus C) \cup (A' \cup B)'.$$

**7. Relations** (4 + 4 + 8 points)

Consider the relations $R$ and $S$, both in the set $V := \{1, 2, 3, 4\}$, that have the directed graph representations given by the following pictures:



$R$



$S$

(a) Is the relation $R$ reflexive? transitive? symmetric? anti-symmetric? Briefly explain your answers.

(b) Is the relation $S$ reflexive? transitive? symmetric? anti-symmetric? Briefly explain your answers.

(c) Explicitly list all the elements of $R \circ S$ in the curly-bracket notation and draw the directed graph representation of $R \circ S$.

**8. Ordering relations** (8 + 3 + 3 points)
Let $A$ be the set defined by

$$A := \{\{\mathtt{b}\}, \{\mathtt{c}\}, \{\mathtt{a}, \mathtt{b}\}, \{\mathtt{b}, \mathtt{c}\}, \{\mathtt{c}, \mathtt{d}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{\mathtt{b}, \mathtt{c}, \mathtt{d}\}, \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}\}.$$

In this task, we consider the partial ordering relation $\subseteq$ on the set $A$.

(a) Use the algorithm that you have learned to construct the Hasse diagram that represents the ordering relation $\subseteq$ on $A$. Explicitly write down the sets $G_x$ and $H_x$ obtained in the construction, and draw the Hasse diagram.

(b) Does the set $A$ have a smallest element according to the ordering relation under consideration? If so, please give this smallest element. If not, please list all the minimal elements of the set $A$.

(c) Does the set $A$ have a largest element according to the ordering relation under consideration? If so, please give this largest element. If not, please list all the maximal elements of the set $A$.