Department of Computer Science
Vrije Universiteit Amsterdam

Midterm Exam for the course *Logic and Sets*
February 27, 2018, 9:00–11:00

This exam has 7 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. Semantic entailment (5 + 5 points)
Argue for each of the following two semantic entailments whether it holds. If so, show this by means of a truth table; if not, give a concrete counterexample.

(a) $(\phi \oplus \psi) \oplus \chi \models \phi \oplus (\psi \oplus \chi)$

(b) $\phi \to (\psi \to \chi) \models (\phi \to \psi) \to \chi$

Solution:

(a)

ϕ	ψ	χ	$\phi \oplus \psi$	$(\phi \oplus \psi) \oplus \chi$	$\psi \oplus \chi$	$\phi \oplus (\psi \oplus \chi)$
T	T	T	F	T	F	T
T	T	F	F	F	T	F
T	F	T	T	F	T	F
T	F	F	T	T	F	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

So the semantic entailment holds, because in every row where the fifth column contains T, the seventh column also contains T. (Actually, the fifth and seventh column coincide, so the two formulas are semantically equivalent.)

(b)

ϕ	ψ	χ	$\psi \to \chi$	$\phi \to (\psi \to \chi)$	$\phi \to \psi$	$(\phi \to \psi) \to \chi$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

So the semantic entailment does not hold, because at the sixth and eighth row there is a T in the fifth column, but an F in the seventh column. (Actually it suffices to only give the sixth or eighth row.)

2. Island puzzle (10 points)

On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth).

You meet three islanders A, B and C.
A says: "if B is a truth speaker, then C is a truth speaker."
B says: "A is a truth speaker or C is a liar." (He means an inclusive or $\vee$.)
C says: "A and B are both liars."

You need to determine, either via a truth table or by means of logical reasoning, which of these three islanders speak the truth and which ones lie.

Also check explicitly that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars.

Solution: The formalizations of the three statements are:

* $t_A \leftrightarrow (t_B \to t_C)$
* $t_B \leftrightarrow (t_A \vee \neg t_C)$
* $t_C \leftrightarrow (\neg t_A \wedge \neg t_B)$

Solution via a truth table:

t_A	t_B	t_C	$t_B \to t_C$	$t_A \leftrightarrow (t_B \to t_C)$	$t_A \vee \neg t_C$	$t_B \leftrightarrow (t_A \vee \neg t_C)$	$\neg t_A \wedge \neg t_B$	$t_C \leftrightarrow (\neg t_A \wedge \neg t_B)$
T	T	T	T	T	T	T	F	F
T	T	F	F	F	T	T	F	T
T	F	T	T	T	T	F	F	F
T	F	F	T	T	T	F	F	T
F	T	T	T	F	F	F	F	F
F	T	F	F	T	T	T	F	T
F	F	T	T	F	F	T	T	T
F	F	F	T	F	T	F	T	F

Only in the sixth row are all three statements awarded T. So A and C are liars while B is a truth speaker.

Solution via logical reasoning: Suppose C is a truth speaker. Then B must be a liar. Then the statement of A is true. So A is a truth speaker. But then the statement of C is false, contradicting the assumption that C is a truth speaker. Hence, C must be a liar.

This means the statement of B is true, so B is a truth speaker.

This means the statement of A is false, so A is a liar.

We check that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars:

- Since B is a truth speaker and C a liar, the statement of A is false.

- Since C is a liar, the statement of B is true.
- Since B is a truth speaker, the statement of C is false.

3. Disjunctive normal form (7 points)
Give the truth table of $(p \vee q) \wedge r$, and use it to construct a formula in DNF that is semantically equivalent.

Solution:

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

Since only the first, third and fifth row of the fifth column contain `T`, the resulting DNF is

$$(p \wedge q \wedge r) \ \vee \ (p \wedge \neg q \wedge r) \ \vee \ (\neg p \wedge q \wedge r)$$

4. Conjunctive normal form (8 points)
Apply the algorithm `CNF` to turn the formula $\neg(p \to q) \vee \neg(\neg p \to \neg q)$ into CNF.

Solution:

$$\begin{aligned}
\neg(p \to q) \vee \neg(\neg p \to \neg q) &\leadsto_{\texttt{IMPL-FREE}} \neg(\neg p \vee q) \vee \neg(\neg p \to \neg q) \\
&\leadsto_{\texttt{IMPL-FREE}} \neg(\neg p \vee q) \vee \neg(\neg\neg p \vee \neg q) \\
&\leadsto_{\texttt{NNF}} (\neg\neg p \wedge \neg q) \vee \neg(\neg\neg p \vee \neg q) \\
&\leadsto_{\texttt{NNF}} (\neg\neg p \wedge \neg q) \vee (\neg\neg\neg p \wedge \neg\neg q) \\
&\leadsto_{\texttt{NNF}} (p \wedge \neg q) \vee (\neg\neg\neg p \wedge \neg\neg q) \\
&\leadsto_{\texttt{NNF}} (p \wedge \neg q) \vee (\neg p \wedge \neg\neg q) \\
&\leadsto_{\texttt{NNF}} (p \wedge \neg q) \vee (\neg p \wedge q) \\
&\leadsto_{\texttt{DISTR}} ((p \wedge \neg q) \vee \neg p) \wedge ((p \wedge \neg q) \vee q) \\
&\leadsto_{\texttt{DISTR}} (p \vee \neg p) \wedge (\neg q \vee \neg p) \wedge ((p \wedge \neg q) \vee q) \\
&\leadsto_{\texttt{DISTR}} (p \vee \neg p) \wedge (\neg q \vee \neg p) \wedge (p \vee q) \wedge (\neg q \vee q)
\end{aligned}$$

(Applying multiple non-overlapping reduction steps at once, like the two $\leadsto_{\texttt{IMPL-FREE}}$ steps at the start, to shorten the derivation, is allowed.)

5. DPLL procedure (10 points)

Apply the DPLL procedure to the CNF $(p \vee q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$, to check whether it is satisfiable.

Solution: First try p is $\top$:

$$(\top \vee q) \wedge (\neg\top \vee q) \wedge (\neg\top \vee \neg q)$$

This reduces to $q \wedge \neg q$. Both attempting $\top$ or $\bot$ for q yields $\bot$, so the temporary verdict is "unsatisfiable".

Now try p is $\bot$:

$$(\bot \vee q) \wedge (\neg\bot \vee q) \wedge (\neg\bot \vee \neg q)$$

This reduces to q. Attempting $\top$ for q yields $\top$. So the final verdict is "satisfiable".

6. Sets (7 + 8 points)

(a) Suppose that in a universe U with 40 elements, we have three sets A, B and C of which we know that

$$\#A = 15 \qquad \#B = 15 \qquad \#C = 15$$
$$\#(A \cap B) = 6 \qquad \#(A \cap C) = 8 \qquad \#(A \cap B \cap C) = 5$$
$$\#(B \setminus C) = 9.$$

Draw a Venn diagram of the sets A, B and C, and determine the number of elements in every region of your Venn diagram. Then use your Venn diagram to determine the numbers

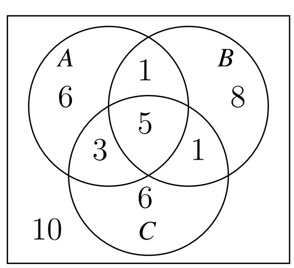
$$\#(A \setminus C) \qquad \text{and} \qquad \#(B \cup C)$$

(b) Derive the following equality (for all sets A, B and C), using the algebra of sets:

$$A \setminus (C \cap B) = (A \setminus C) \cup (A' \cup B)'.$$

Solution:

(a) This is the Venn diagram with the number of elements for each region:

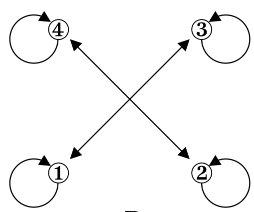


(b) We see from the Venn diagram that

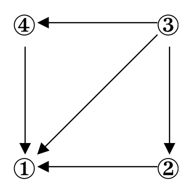
$$\#(A \setminus C) = 7 \qquad \text{and} \qquad \#(B \cup C) = 24.$$

(c)
$$\begin{aligned} A \setminus (C \cap B) &= A \cap (C \cap B)' && \text{(definition)} \\ &= A \cap (C' \cup B') && \text{(De Morgan)} \\ &= (A \cap C') \cup (A \cap B') && \text{(distributivity)} \\ &= (A \setminus C) \cup (A \cap B') && \text{(definition)} \\ &= (A \setminus C) \cup ((A')' \cap B') && \text{(involution)} \\ &= (A \setminus C) \cup (A' \cup B)' && \text{(De Morgan).} \end{aligned}$$

7. Relations (4 + 4 + 8 points)
Consider the relations R and S, both in the set $V := \{1, 2, 3, 4\}$, that have the directed graph representations given by the following pictures:



R



S

(a) Is the relation R reflexive? transitive? symmetric? anti-symmetric? Briefly explain your answers.

(b) Is the relation S reflexive? transitive? symmetric? anti-symmetric? Briefly explain your answers.

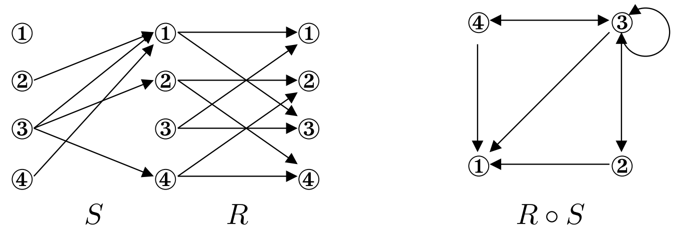
(c) Explicitly list all the elements of $R \circ S$ in the curly-bracket notation and draw the directed graph representation of $R \circ S$.

Solution:

(a) Reflexive: yes, every element relates to itself.
Transitive: yes, since if you pick any two elements x and z (e.g. 1 and 2) such that $\neg(x\,R\,z)$, then there is actually no path from x to z at all, so in particular, there is no path that goes from x to z via a single intermediate point y.
Symmetric: yes, whenever an element x relates to y, y also relates to x.
Anti-symmetric: no, since $1\,R\,3$ and $3\,R\,1$.

(b) Reflexive: no, no element relates to itself.
Transitive: yes, the only two two-step paths are from 3 via 4 to 1, and from 3 via 2 to 1. But 3 also relates to 1 directly.
Symmetric: no, since $3\,R\,1$ but $\neg(1\,R\,3)$.
Anti-symmetric: yes, all relations between two different points go only in one direction.

(c) We draw a Venn representation of the composition $R \circ S$:



From the Venn representation we can read off all the pairs that together make up the composition:

$$R \circ S = \{\langle 2,1\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle, \langle 4,3\rangle\}$$

We can now draw the directed graph representation. See again the picture above, on the right.

8. Ordering relations (8 + 3 + 3 points)
Let A be the set defined by

$$A := \{\{\mathtt{b}\}, \{\mathtt{c}\}, \{\mathtt{a},\mathtt{b}\}, \{\mathtt{b},\mathtt{c}\}, \{\mathtt{c},\mathtt{d}\}, \{\mathtt{a},\mathtt{b},\mathtt{c}\}, \{\mathtt{b},\mathtt{c},\mathtt{d}\}, \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\}.$$

In this task, we consider the partial ordering relation $\subseteq$ on the set A.

(a) Use the algorithm that you have learned to construct the Hasse diagram that represents the ordering relation $\subseteq$ on A. Explicitly write down the sets G_x and H_x obtained in the construction, and draw the Hasse diagram.

(b) Does the set A have a smallest element according to the ordering relation under consideration? If so, please give this smallest element. If not, please list all the minimal elements of the set A.

(c) Does the set A have a largest element according to the ordering relation under consideration? If so, please give this largest element. If not, please list all the maximal elements of the set A.

Solution:

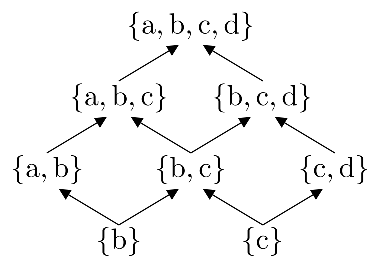
(a) The algorithm gives the following sets G_x and H_x:

$$\begin{aligned}
G_{\{\mathtt{b}\}} &= \{\{\mathtt{a},\mathtt{b}\}, \{\mathtt{b},\mathtt{c}\}, \{\mathtt{a},\mathtt{b},\mathtt{c}\}, \{\mathtt{b},\mathtt{c},\mathtt{d}\}, \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\} \\
G_{\{\mathtt{c}\}} &= \{\{\mathtt{b},\mathtt{c}\}, \{\mathtt{c},\mathtt{d}\}, \{\mathtt{a},\mathtt{b},\mathtt{c}\}, \{\mathtt{b},\mathtt{c},\mathtt{d}\}, \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\} \\
G_{\{\mathtt{a},\mathtt{b}\}} &= \{\{\mathtt{a},\mathtt{b},\mathtt{c}\}, \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\} \\
G_{\{\mathtt{b},\mathtt{c}\}} &= \{\{\mathtt{a},\mathtt{b},\mathtt{c}\}, \{\mathtt{b},\mathtt{c},\mathtt{d}\}, \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\} \\
G_{\{\mathtt{c},\mathtt{d}\}} &= \{\{\mathtt{b},\mathtt{c},\mathtt{d}\}, \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\} \\
G_{\{\mathtt{a},\mathtt{b},\mathtt{c}\}} &= \{\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\} \\
G_{\{\mathtt{b},\mathtt{c},\mathtt{d}\}} &= \{\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}\} \\
G_{\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}} &= \{\,\}
\end{aligned}$$

and

$$
\begin{aligned}
H_{\{\mathtt{b}\}} &= \{\{\mathtt{a}, \mathtt{b}\}, \{\mathtt{b}, \mathtt{c}\}\} \\
H_{\{\mathtt{c}\}} &= \{\{\mathtt{b}, \mathtt{c}\}, \{\mathtt{c}, \mathtt{d}\}\} \\
H_{\{\mathtt{a},\mathtt{b}\}} &= \{\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}\} \\
H_{\{\mathtt{b},\mathtt{c}\}} &= \{\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{\mathtt{b}, \mathtt{c}, \mathtt{d}\} \\
H_{\{\mathtt{c},\mathtt{d}\}} &= \{\{\mathtt{b}, \mathtt{c}, \mathtt{d}\}\} \\
H_{\{\mathtt{a},\mathtt{b},\mathtt{c}\}} &= \{\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}\} \\
H_{\{\mathtt{b},\mathtt{c},\mathtt{d}\}} &= \{\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}\} \\
H_{\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}} &= \{\,\}
\end{aligned}
$$

The Hasse diagram therefore looks like this:



(b) As the Hasse diagram shows, there is no smallest element. There are two minimal elements: $\{\mathtt{b}\}$ and $\{\mathtt{c}\}$.

(c) As we see in the Hasse diagram, there is a largest element, namely $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$.