This exam has 3 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. Semantic entailment / CNF (7+5 points)

(a) Decide for each of the following two statements, by means of a truth table, whether it is a valid semantic entailment:

- $(p \to q) \to r \models p \to (q \to r)$
- $p \to (q \to r) \models (p \to q) \to r$

(b) Use the CNF algorithm to reduce $(p \to q) \to r$ to conjunctive normal form.

2. DNF / Logic circuits (5+8 points)
Consider the Boolean function f defined by: $f(x, y, z) = 0$ if exactly one of the three arguments x, y and z is 0 (and the other two are 1).

(a) Give the truth table of f, and use it to construct a disjunctive normal form that represents f.

(b) Construct a logic circuit consisting only of *and*, *or* and *not* gates that represents f (with three input channels, for x, y and z, and one output channel, for $f(x, y, z)$).

3. OBDDs (3+8 points)

(a) Represent $(x \land y) \oplus z$ by means of a binary decision tree, with respect to the variable ordering x, y, z.

(b) Reduce this binary decision tree to an ordered binary decision diagram. (Also give the intermediate reduction steps.)

4. Models for predicate logic (9 points)
Give a model, consisting of two elements, that shows the two predicate logic formulas below are not equivalent.

- $\forall x\, (C(x) \lor D(x))$
- $(\forall x\, C(x)) \lor (\forall x\, D(x))$

5. Sets (5 + 6 points)
In this task, we consider the following set-theoretic equality:

$$B \setminus (A \cup C)' = (B \setminus A') \cup (B \setminus C')$$

(a) Draw Venn diagrams for the two sets $B \setminus (A \cup C)'$ and $(B \setminus A') \cup (B \setminus C')$, depicting clearly which area corresponds to the set, and how the set is constructed from A, B and C using the fundamental set operations. Use your Venn diagrams to conclude that the set-theoretic equality displayed above holds.

(b) Now prove the set-theoretic equality displayed above (for all sets A, B and C) using the algebra of sets (NB: the laws of the algebra of sets are on the last page of this exam).

6. Ordering relations (6+2+2 points)
In this task, we consider the *strict* ordering relation $<_A$ on the set $A := \{1, 2, 3, 4\}^2$ defined by the description

$$\langle a, b\rangle <_A \langle c, d\rangle \qquad \Longleftrightarrow \qquad a + b < c + d.$$

Here, $<$ denotes the usual strict ordering relation on the integers. Let $\leq_A$ be the *partial* order on A corresponding to the strict order $<_A$ introduced above. We now restrict our attention to the subset B of A defined by

$$B := \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle\}.$$

(a) Use the algorithm you have learned to construct the Hasse diagram of the partial ordering relation $\leq_A$ on the set B. Explicitly write down the sets G_x and H_x obtained in the construction, and draw the Hasse diagram. (To simplify the notation, you can write 23 for $\langle 2, 3\rangle$, 31 for $\langle 3, 1\rangle$, and so on.)

(b) Does the set B have a smallest element according to the relation $\leq_A$? If so, please give this smallest element. If not, please list all the minimal elements of B.

(c) Does the set B have a largest element according to the relation $\leq_A$? If so, please give this largest element. If not, please list all the maximal elements of B.

7. Functions and equivalence relations (4 + 6 + 4 points)
At the end of this academic year, more than 100 students will have received a final grade for the course *Logic and Sets*. Let *Students* be the set of all students who actively participated in the course, and let *Grades* be the set of possible final grades, that is

$$\mathit{Grades} = \{1, 1.5, 2, 2.5, \ldots, 10\} \setminus \{5.5\}.$$

At the end of the academic year, we can construct the relation *HasFinalGrade* of type *Students* × *Grades*, defined by

$$\mathit{HasFinalGrade} := \{\langle x, y\rangle : x \text{ has final grade } y\}.$$

(a) For each of the relations *HasFinalGrade* and $\mathit{HasFinalGrade}^{-1}$, decide whether it is a function or not. Please briefly explain your answers.

(b) Next, consider the relation $R := \mathit{HasFinalGrade}^{-1} \circ \mathit{HasFinalGrade}$. Please show that R is an equivalence relation.

(c) Describe in words what the partition into equivalence classes induced by R looks like (that is, what do elements in the same equivalence class have in common, and how do they differ from elements in a different equivalence class?).

8. Induction (10 points)

In this task, we consider a sequence $(t_n)_{n=1}^{\infty}$ of integer numbers defined recursively by

$$t_1 := 3, \qquad t_{n+1} := t_n + 6n + 3.$$

We claim that for all $n \geq 1$,

$$t_n = 3n^2.$$

Prove this claim by mathematical induction.