Department of Computer Science
VU University Amsterdam

Resit Exam for the course *Logic and Sets*
June 15, 2017, 18:30–21:15

This exam has 8 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. Semantic entailment / CNF (7+5 points)

(a) Decide for each of the following two statements, by means of a truth table, whether it is a valid semantic entailment:

- $(p \to q) \to r \models p \to (q \to r)$
- $p \to (q \to r) \models (p \to q) \to r$

(b) Use the CNF algorithm to reduce $(p \to q) \to r$ to conjunctive normal form.

Solution:

(a)

p	q	r	$p \to q$	$q \to r$	$(p \to q) \to r$	$p \to (q \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

So the first semantic entailment holds, because at every row where there is a T in the sixth column, there is also a T in the seventh column.

However, the second semantic entailment does not hold, because at the sixth and eighth row there is a T in the seventh column, but an F in the sixth column.

(b) $(p \to q) \to r \rightsquigarrow_{\texttt{IMPL-FREE}} (\neg p \vee q) \to r \rightsquigarrow_{\texttt{IMPL-FREE}} \neg(\neg p \vee q) \vee r \rightsquigarrow_{\texttt{NNF}} (\neg\neg p \wedge \neg q) \vee r \rightsquigarrow_{\texttt{NNF}} (p \wedge \neg q) \vee r \rightsquigarrow_{\texttt{DISTR}} (p \vee r) \wedge (\neg q \vee r)$.

2. DNF / Logic circuits (5+8 points)

Consider the Boolean function f defined by: $f(x, y, z) = 0$ if exactly one of the three arguments x, y and z is 0 (and the other two are 1).

(a) Give the truth table of f, and use it to construct a disjunctive normal form that represents f.

(b) Construct a logic circuit consisting only of *and*, *or* and *not* gates that represents f (with three input channels, for x, y and z, and one output channel, for $f(x, y, z)$).

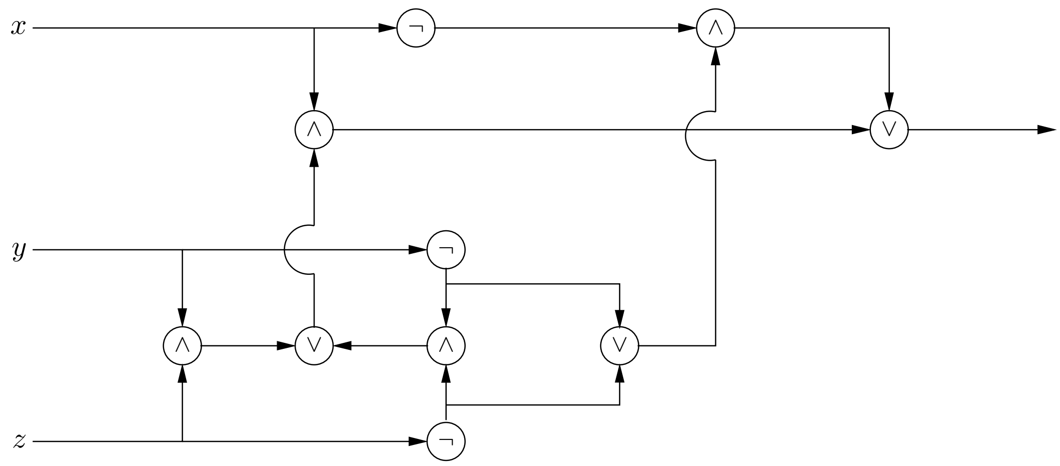
Solution:

(a)

x	y	z	$f(x,y,z)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

$$(x \wedge y \wedge z) \vee (\neg x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge \neg z) \vee (x \wedge \neg y \wedge \neg z) \vee (\neg x \wedge \neg y \wedge \neg z).$$

(b) It is more convenient to build a circuit for the equivalent formula $(x \wedge ((y \wedge z) \vee (\neg y \wedge \neg z))) \vee (\neg x \wedge (\neg y \vee \neg z))$ (than directly representing the DNF above).



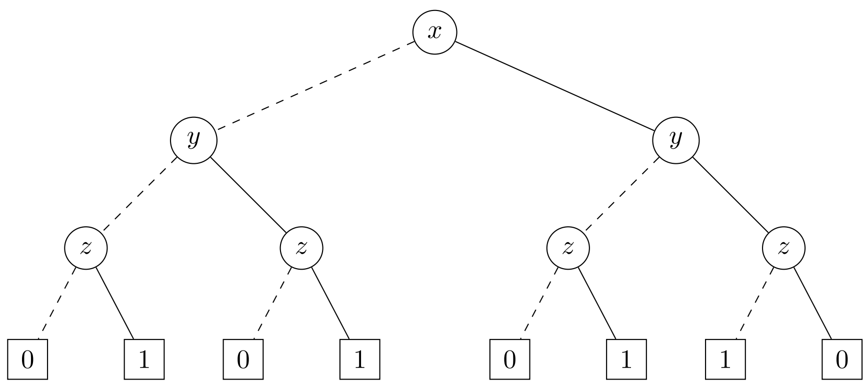
3. OBDDs (3+8 points)

(a) Represent $(x \wedge y) \oplus z$ by means of a binary decision tree, with respect to the variable ordering x, y, z.

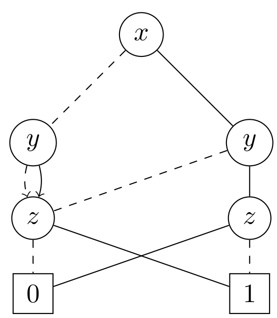
(b) Reduce this binary decision tree to an ordered binary decision diagram. (Also give the intermediate reduction steps.)

Solution:

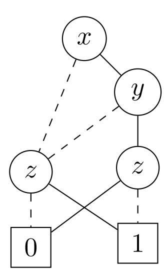
(a)



(b) Apply C1 twice, to collapse leaves, yields



Apply C2 to remove the left-hand node y:



4. **Models for predicate logic** (9 points)
Give a model, consisting of two elements, that shows the two predicate logic formulas below are not equivalent.

- $\forall x\,(C(x) \vee D(x))$
- $(\forall x\, C(x)) \vee (\forall x\, D(x))$

Solution: The two formulas are not equivalent. Consider a model consisting of a set of two elements $\{a, b\}$, where the predicate C holds only for a while the predicate D holds only for b.

The first formula holds on this model: on a, C or D holds (namely, C), and also on b, C or D holds (namely, D). However, the second formula does not hold on this model: $\forall x\, C(x)$ fails because C does not hold for b, while $\forall x\, D(x)$ fails because D does not hold for a; so the disjunction of these two subformulas also does not hold on the model.

5. **Sets** (5 + 6 points)
In this task, we consider the following set-theoretic equality:

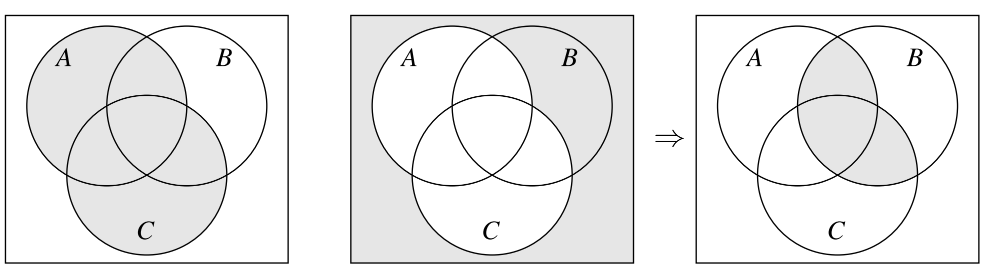
$$B \setminus (A \cup C)' = (B \setminus A') \cup (B \setminus C')$$

(a) Draw Venn diagrams for the two sets $B \setminus (A \cup C)'$ and $(B \setminus A') \cup (B \setminus C')$, depicting clearly which area corresponds to the set, and how the set is constructed from A, B and C using the fundamental set operations. Use your Venn diagrams to conclude that the set-theoretic equality displayed above holds.

(b) Now prove the set-theoretic equality displayed above (for all sets A, B and C) using the algebra of sets (NB: the laws of the algebra of sets are on the last page of this exam).

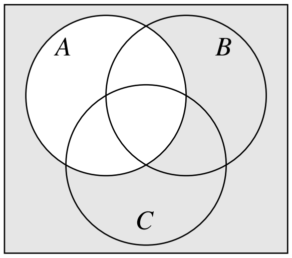
Solution:

(a) The following Venn diagrams depict the construction of the set $B \setminus (A \cup C)'$:



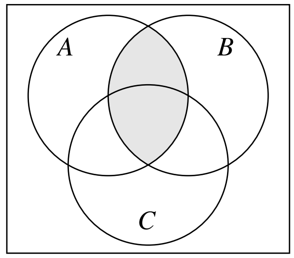
$A \cup C$ $\quad$ $(A \cup C)'$ $\quad$ $B \setminus (A \cup C)'$

These Venn diagrams depict the construction of $(B \setminus A') \cup (B \setminus C')$:

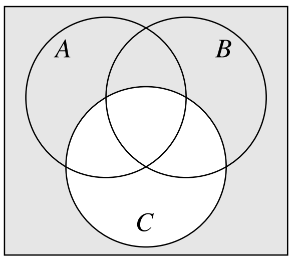


A'

$\Rightarrow$

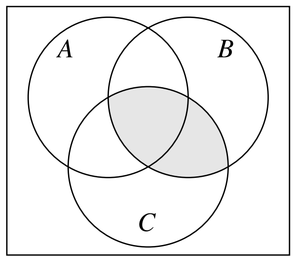


$B \setminus A'$



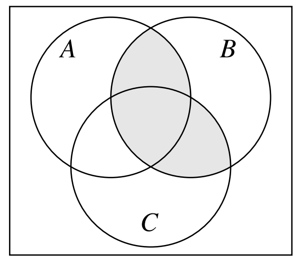
C'

$\Rightarrow$



$B \setminus C'$

$\Rightarrow$



$(B \setminus A') \cup (B \setminus C')$

As we can see, the shaded area in the Venn-diagram of $B \setminus (A \cup C)'$ corresponds exactly to the shaded area in the Venn-diagram of $(B \setminus A') \cup (B \setminus C')$. Therefore,

$$B \setminus (A \cup C)' = (B \setminus A') \cup (B \setminus C')$$

(b) We now derive the desired equality using the algebra of sets:

$$\begin{aligned}
B \setminus (A \cup C)' &= B \cap ((A \cup C)')' && \text{(definition)} \\
&= B \cap (A \cup C) && \text{(involution)} \\
&= (B \cap A) \cup (B \cap C) && \text{(distributivity)} \\
&= (B \cap (A')') \cup (B \cap (C')') && \text{(involution)} \\
&= (B \setminus A') \cup (B \setminus C') && \text{definition).}
\end{aligned}$$

6. Ordering relations (6+2+2 points)
In this task, we consider the *strict* ordering relation $<_A$ on the set $A := \{1, 2, 3, 4\}^2$ defined by the description

$$\langle a, b \rangle <_A \langle c, d \rangle \qquad \iff \qquad a + b < c + d.$$

Here, $<$ denotes the usual strict ordering relation on the integers. Let $\leq_A$ be the *partial* order on A corresponding to the strict order $<_A$ introduced above. We now restrict our attention to the subset B of A defined by

$$B := \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle\}.$$

(a) Use the algorithm you have learned to construct the Hasse diagram of the partial ordering relation $\leq_A$ on the set B. Explicitly write down the sets G_x and H_x obtained in the construction, and draw the Hasse diagram. (To simplify the notation, you can write 23 for $\langle 2,3\rangle$, 31 for $\langle 3,1\rangle$, and so on.)

(b) Does the set B have a smallest element according to the relation $\leq_A$? If so, please give this smallest element. If not, please list all the minimal elements of B.

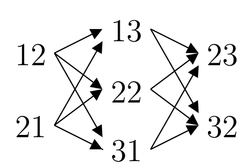
(c) Does the set B have a largest element according to the relation $\leq_A$? If so, please give this largest element. If not, please list all the maximal elements of B.

Solution:

(a) The algorithm leads to the following sets G_x and H_x:

$G_{12} = \{13, 22, 23, 31, 32\}$	$H_{12} = \{13, 22, 31\}$
$G_{13} = \{23, 32\}$	$H_{13} = \{23, 32\}$
$G_{21} = \{13, 22, 23, 31, 32\}$	$H_{21} = \{23, 32\}$
$G_{22} = \{23, 32\}$	$H_{22} = \{23, 32\}$
$G_{23} = \varnothing$	$H_{23} = \varnothing$
$G_{31} = \{23, 32\}$	$H_{31} = \{23, 32\}$
$G_{32} = \varnothing$	$H_{32} = \varnothing$

The Hasse diagram therefore looks like this:



(b) The set B does not have a smallest element; the two minimal elements of B are $\langle 1,2\rangle$ and $\langle 2,1\rangle$.

(c) The set B does not have a largest element either; the two maximal elements of B are $\langle 2,3\rangle$ and $\langle 3,2\rangle$.

7. Functions and equivalence relations (4 + 6 + 4 points)
At the end of this academic year, more than 100 students will have received a final grade for the course *Logic and Sets*. Let *Students* be the set of all students who actively participated in the course, and let *Grades* be the set of possible final grades, that is

$$\mathit{Grades} = \{1, 1.5, 2, 2.5, \ldots, 10\} \setminus \{5.5\}.$$

At the end of the academic year, we can construct the relation *HasFinalGrade* of type *Students* $\times$ *Grades*, defined by

$$HasFinalGrade := \{\langle x, y\rangle : x \text{ has final grade } y\}.$$

(a) For each of the relations *HasFinalGrade* and $HasFinalGrade^{-1}$, decide whether it is a function or not. Please briefly explain your answers.

(b) Next, consider the relation $R := HasFinalGrade^{-1} \circ HasFinalGrade$. Please show that R is an equivalence relation.

(c) Describe in words what the partition into equivalence classes induced by R looks like (that is, what do elements in the same equivalence class have in common, and how do they differ from elements in a different equivalence class?).

Solution:

(a) Since every student is assigned *exactly one* final grade, *HasFinalGrade* is a function. On the other hand, there must be at least one grade that is obtained by more than one student (i.e., the function *HasFinalGrade* is not injective). Therefore, $HasFinalGrade^{-1}$ is not a function.

(b) To show that R is an equivalence relation, we have to show that R is reflexive, symmetric, and transitive. To simplify the notation, let us abbreviate the relation *HasFinalGrade* to *HFG*.

  (i) Reflexivity: every student receives a final grade, so for every $x \in Students$, there is a $y \in Grades$ such that $x\ HFG\ y$. But then also $y\ HFG^{-1}\ x$, hence by the definition of composition of relations, $x\ R\ x$.

  (ii) Symmetry: suppose that $x\ R\ y$. Then, by the definition of composition of relations, we know that there exists a grade z such that $x\ HFG\ z$ and $z\ HFG^{-1}\ y$. But then we also have that $y\ HFG\ z$ and $z\ HFG^{-1}\ x$, and therefore $y\ R\ z$.

  (iii) Transitivity: suppose that $x\ R\ y$ and $y\ R\ z$. Then we know that there are grades u and v such that $x\ HFG\ u$, $u\ HFG^{-1}\ y$, $y\ HFG\ v$ and $v\ HFG^{-1}\ z$. In particular, it holds that $y\ HFG\ u$ and $y\ HFG\ v$. But since every student receives exactly one final grade, it must be the case that $u = v$. It follows that $x\ HFG\ u$ and $u\ HFG^{-1}\ z$, and therefore, $x\ R\ z$.

(c) R partitions the set of students into equivalence classes in such a way, that students in the same equivalence class all have the same final grade, and students in different equivalence classes have different grades.

8. Induction (10 points)

In this task, we consider a sequence $(t_n)_{n=1}^{\infty}$ of integer numbers defined recursively by

$$t_1 := 3, \qquad t_{n+1} := t_n + 6n + 3.$$

We claim that for all $n \geq 1$,

$$t_n = 3n^2.$$

Prove this claim by mathematical induction.

Solution: Base case ($n = 1$):

$$t_1 = 3 = 3 \cdot 1^2. \qquad \checkmark$$

Let $m \geq 1$ be arbitrary. We assume, as our *inductive hypothesis*, that

$$t_m = 3m^2.$$

As our *inductive step*, we now derive that

$$\begin{aligned} t_{m+1} &= t_m + 6m + 3 && \text{(by the recursive definition)} \\ &= 3m^2 + 6m + 3 && \text{(by the inductive hypothesis)} \\ &= 3(m^2 + 2m + 1) && \text{(arithmetic)} \\ &= 3(m+1)^2 && \text{(arithmetic)} \qquad \checkmark \end{aligned}$$

This is exactly the statement in the claim for $n = m + 1$, which completes the proof.