Department of Computer Science	Final Exam for the course *Logic and Sets*
VU University Amsterdam	March 30, 2017, 08:45–11:30

This exam has 3 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. Propositional logic (2+3+6+4 points)
Consider the propositional formula $\neg(p \wedge \neg q) \wedge (p \vee r)$.

(a) Provide the truth table for this formula.

(b) Use the truth table you constructed in (a) to turn this formula into a DNF.

(c) Use the laws for semantic equivalence in propositional logic to derive that the formula above and the DNF you constructed in (b) are semantically equivalent.

(d) Construct a logic circuit that corresponds to $\neg(p \wedge \neg q) \wedge (p \vee r)$.

2. Island puzzle (8 points)
On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth).

You meet three islanders A, B and C.
A says: "B or C is a liar."
B says: "C is a liar."
C says: "A is a liar."

You need to determine, *by means of logical reasoning using propositional formulas*, which of these three islanders speak the truth and which ones lie. Explain your argumentation. Also check explicitly that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars.

3. OBDDs (5+6 points)
Consider the variable order x, y, z. Give all steps in the following constructions.

(a) Construct the reduced OBDD for the propositional formula $(x \wedge y) \vee (\neg x \wedge \neg z)$ from its binary decision tree.

(b) Construct OBBDs for $\forall x\,((x \wedge y) \vee (\neg x \wedge \neg z))$ and $\exists x\,((x \wedge y) \vee (\neg x \wedge \neg z))$, using the OBDD you constructed in (a).

4. **Models for predicate logic** (7+4 points)
For each of the following pairs of formulas, either argue that they are semantically equivalent, or give a model on which their truth values are different.

(a) $\exists x \, \forall y \, R(x, y)$ and $\exists x \, \forall y \, R(y, x)$

(b) $\exists x \, \exists y \, R(x, y)$ and $\exists x \, \exists y \, R(y, x)$

5. **Sets** (5 + 5 points) In this task, we consider the following set-theoretic equality:

$$(C' \cup (A \cap B))' = (C \setminus A) \cup (C \setminus B)$$

(a) Draw Venn diagrams for the two sets $C' \cup (A \cap B)$ and $(C \setminus A) \cup (C \setminus B)$, depicting clearly which area corresponds to the set, and how the set is constructed from A, B and C using the fundamental set operations. Use your Venn diagrams to conclude that the set-theoretic equality displayed above holds.

(b) Now prove the set-theoretic equality displayed above (for all sets A, B and C) using the algebra of sets (NB: the laws of the algebra of sets are on the last page of this exam).

6. **Binary relations** (4 + 4 + 5 points)
This task is about binary relations in the set $A := \{1, 2, 3, 6\}^2$.

(a) First we consider the equivalence relation $\equiv$ on A defined by the description

$$\langle a, b \rangle \equiv \langle c, d \rangle \qquad \Longleftrightarrow \qquad a \cdot b = c \cdot d.$$

Explicitly write down all equivalence classes (using the curly-bracket notation), and give a complete system of representatives for this equivalence relation.

(b) Next, we define a *strict* ordering relation $<$ on the set A by the description

$$\langle a, b \rangle < \langle c, d \rangle \qquad \Longleftrightarrow \qquad a \cdot b < c \cdot d.$$

Show that this relation is antisymmetric and transitive.

(c) Let $\leq$ be the *partial* order on A corresponding to the strict order $<$ introduced above. We now restrict our attention to the subset $B = \{2, 3, 6\}^2$ of A, that is,

$$B = \{\langle 2, 2 \rangle, \langle 2, 3 \rangle, \langle 2, 6 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle, \langle 3, 6 \rangle, \langle 6, 2 \rangle, \langle 6, 3 \rangle, \langle 6, 6 \rangle\}.$$

Use the algorithm you have learned to construct the Hasse diagram of the partial ordering relation $\leq$ on the set B. Explicitly write down the sets G_x and H_x obtained in the construction, and draw the Hasse diagram. (To simplify the notation, you can write 23 for $\langle 2, 3 \rangle$, 33 for $\langle 3, 3 \rangle$, and so on.)

7. Functions (4 + 4 + 4 points)

(a) In the set of all *People*, consider the two relations *IsFatherOf* and *IsOnlyChildOf* (where *Father* and *Child* are to be interpreted in the biological sense). For each of the relations *IsFatherOf*, *IsOnlyChildOf*, $\mathit{IsFatherOf}^{-1}$ and $\mathit{IsOnlyChildOf}^{-1}$, determine whether or not it is a function. Explain your answers.

For the remainder of this exercise, we work with the functions $sqr\colon \{x\colon x \geq 0\} \to \mathrm{R}$ and $sub\colon \mathrm{R} \to \mathrm{R}$ defined by

$$sqr(x) = x^2 \qquad\qquad sub(x) = x - 1.$$

(b) Give an explicit description (i.e. a formula for $f(x)$) of the function f given by

$$f := sub^{-1} \circ sqr^{-1} \circ sub,$$

and write down the domain of definition and range (image) of f.

(c) Express the function $g\colon \mathrm{R} \to \mathrm{R}$ defined by

$$g(x) := \sqrt{x^2 + 1}.$$

as a composition of the functions sqr, sub and their inverses.

8. Induction (10 points)

We claim that for all $n \in \mathrm{N}$

$$\sum_{k=0}^{n} 3^k = \frac{1}{2}(3^{n+1} - 1).$$

Your task is to prove this claim by mathematical induction.

NB: if we write out the sum, the claim is that

$$3^0 + 3^1 + 3^2 + \cdots + 3^n = \tfrac{1}{2}(3^{n+1} - 1) \qquad \text{for all } n \in \mathrm{N}.$$