This exam has 9 pages and 8 exercises.

The result will be computed as (total number of points plus 10) divided by 10.

Answers may be given in either English or Dutch.

Please motivate all answers!

1. Propositional logic (2+3+6+4 points)
Consider the propositional formula $\neg(p \wedge \neg q) \wedge (p \vee r)$.

(a) Provide the truth table for this formula.

(b) Use the truth table you constructed in (a) to turn this formula into a DNF.

(c) Use the laws for semantic equivalence in propositional logic to derive that the formula above and the DNF you constructed in (b) are semantically equivalent.

(d) Construct a logic circuit that corresponds to $\neg(p \wedge \neg q) \wedge (p \vee r)$.

Solution:

(a)

p	q	r	$\neg(p \wedge \neg q) \wedge (p \vee r)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

(b) Four lines in the truth table have the value T for the formula. This yields the DNF

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$$

(c) $\neg(p \wedge \neg q) \wedge (p \vee r)$

$\equiv (\neg p \vee \neg\neg q) \wedge (p \vee r)$ (De Morgan)

$\equiv (\neg p \vee q) \wedge (p \vee r)$ (involution)

$\equiv (\neg p \wedge p) \vee (\neg p \wedge r) \vee (q \wedge p) \vee (q \wedge r)$ (distributivity)

$\equiv \bot \vee (\neg p \wedge r) \vee (p \wedge q) \vee (q \wedge r)$ (complement+commutativity)

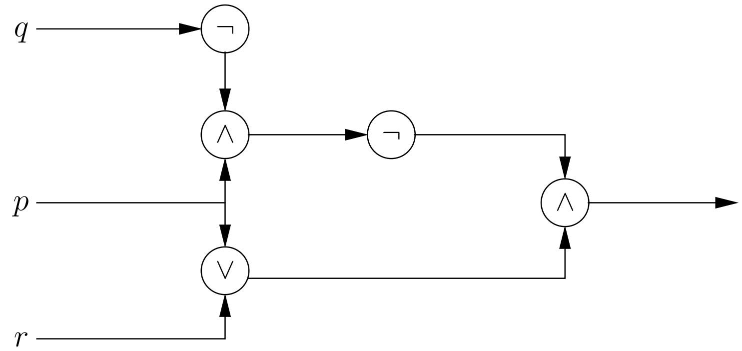
$\equiv (\neg p \wedge \top \wedge r) \vee (p \wedge q \wedge \top) \vee (\top \wedge q \wedge r)$ (identity)

$\equiv (\neg p \wedge (q \vee \neg q) \wedge r) \vee (p \wedge q \wedge (r \vee \neg r)) \vee ((p \vee \neg p) \wedge q \wedge r)$ (complement)

$\equiv (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r)$ (distributivity)

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$ (idempotence+commutativity)

(d)



2. Island puzzle (8 points)
On the island of liars and truth speakers, everybody is either a liar (who always lies) or a truth speaker (who always speaks the truth).

You meet three islanders A, B and C.
A says: "B or C is a liar."
B says: "C is a liar."
C says: "A is a liar."

You need to determine, *by means of logical reasoning using propositional formulas*, which of these three islanders speak the truth and which ones lie. Explain your argumentation. Also check explicitly that the claims by the three islanders are in line with your conclusion whether they are truth speakers or liars.

Solution: By the claim of A we have: $t_A \leftrightarrow (\neg t_B \vee \neg t_C)$. By the claim of B we have: $t_B \leftrightarrow \neg t_C$. By the claim of C we have: $t_C \leftrightarrow \neg t_A$.

Assume, toward a contradiction, that B is a liar: t_B is *false*. Then by the claim of B, C is a truth speaker: t_C is *true*. So by the claim of C, A is a liar: t_A is *false*. So by the claim of A, $\neg t_B \vee \neg t_C$ is *false*, meaning that t_B and t_C are both *true*. But we assumed that t_B is *false*. Contradiction.

So the assumption cannot hold, and hence B is a truth speaker: t_B is *true*. Then by the claim of B, C is a liar: t_C is *false*. So by the claim of C, A is a truth speaker: t_A is *true*.

Check: Since C is a liar, t_C is *false*, so $\neg t_B \vee \neg t_C$ is *true*; so the claims by A and B are true. Since A is a truth speaker, t_A is *true*, so the claim by C is a lie.

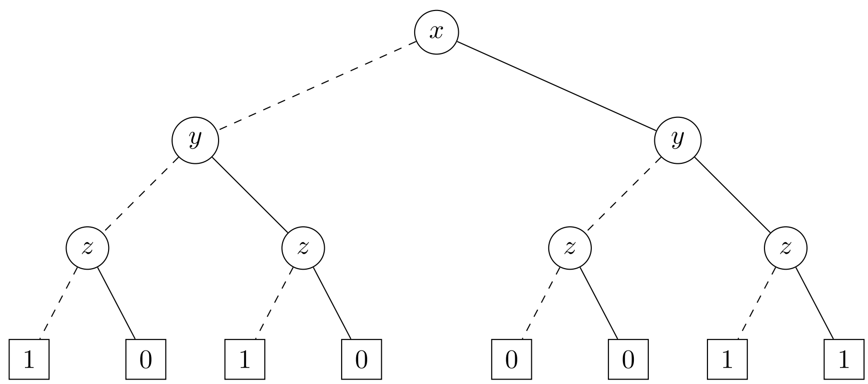
3. OBDDs (5+6 points)
Consider the variable order x, y, z. Give all steps in the following constructions.

(a) Construct the reduced OBDD for the propositional formula $(x \wedge y) \vee (\neg x \wedge \neg z)$ from its binary decision tree.

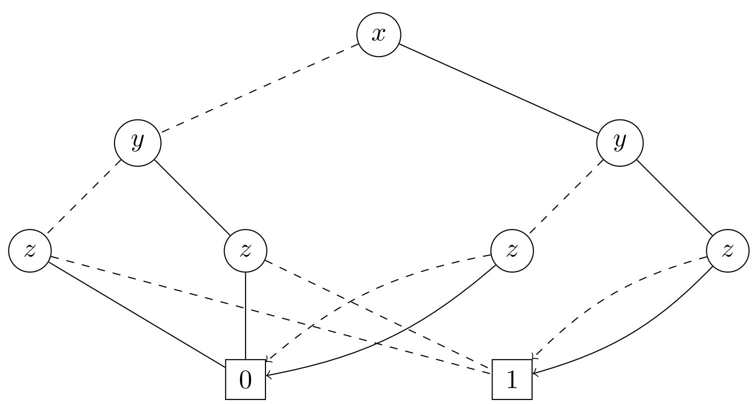
(b) Construct OBBDs for $\forall x\,((x \wedge y) \vee (\neg x \wedge \neg z))$ and $\exists x\,((x \wedge y) \vee (\neg x \wedge \neg z))$, using the OBDD you constructed in (a).

Solution:

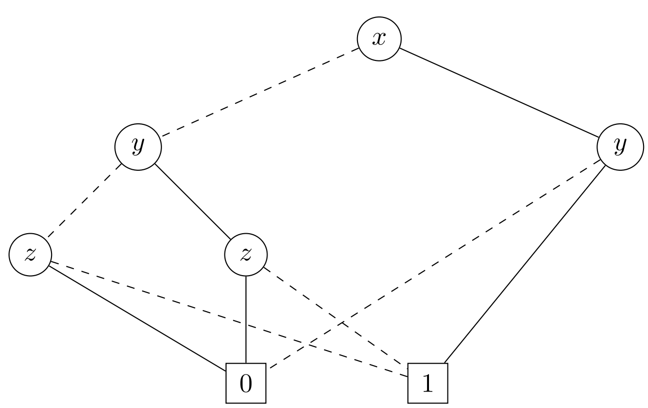
(a)



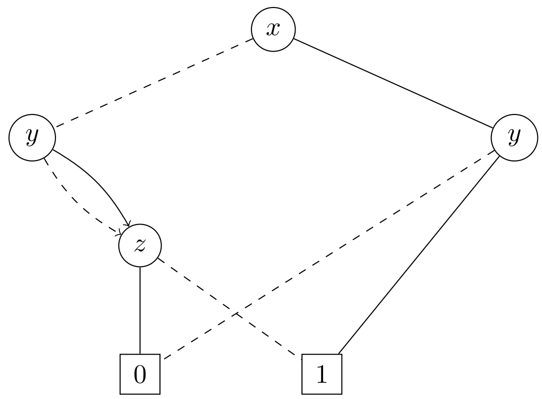
Collapse leaves (C1).



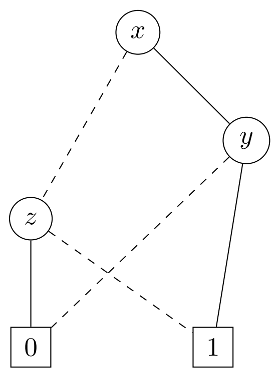
Eliminate nodes of which the 0- and 1-transition lead to the same node (C2, two times).



Collapse nodes that have identical 0-and 1-transitions.

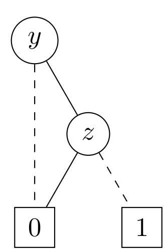


Eliminate nodes of which the 0- and 1-transition lead to the same node (C2).

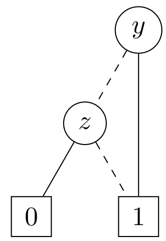


(b) Taking $x = 1$ and $x = 0$, we get the following two OBDDs, respectively:

To construct the OBDD for $\forall x\,((x \wedge y) \vee (\neg x \wedge \neg z))$, we must compute the conjunction of these two OBDDs:



To construct the OBDD for $\exists x\,((x \wedge y) \vee (\neg x \wedge \neg z))$, we must compute the disjunction of these two OBDDs:



4. Models for predicate logic (7+4 points)
For each of the following pairs of formulas, either argue that they are semantically equivalent, or give a model on which their truth values are different.

(a) $\exists x\, \forall y\, R(x,y)$ and $\exists x\, \forall y\, R(y,x)$

(b) $\exists x\, \exists y\, R(x,y)$ and $\exists x\, \exists y\, R(y,x)$

Solution:

(a) These formulas are not semantically equivalent.

Let the model consist of the set $\{a,b\}$ and relations $R(a,a)$ and $R(a,b)$. Then the first formula holds (take $x = a$). But the second does not, because $R(b,a)$ and $R(b,b)$ do not hold.

Alternatively, one can build a model where the second formula holds but the first does not: take as relations $R(a,a)$ and $R(b,a)$.

(b) These formulas are semantically equivalent. In any model $\mathcal{M}$, $R(x,y)$ holds for some $x = a$ and $y = b$, where a, b are elements of the set belonging to $\mathcal{M}$, if and only if $R(y,x)$ holds for $y = a$ and $x = b$.

5. Sets (5 + 5 points) In this task, we consider the following set-theoretic equality:

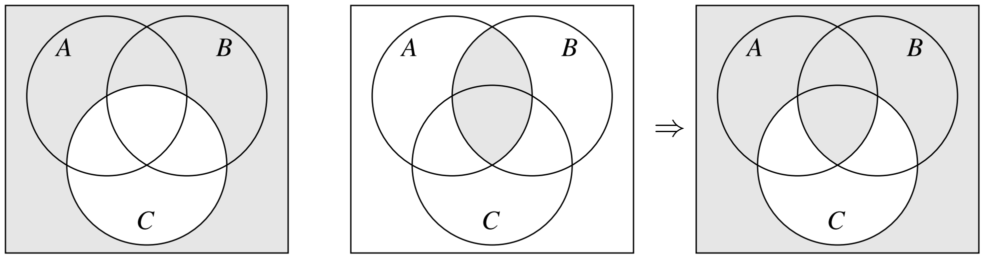
$$(C' \cup (A \cap B))' = (C \setminus A) \cup (C \setminus B)$$

(a) Draw Venn diagrams for the two sets $C' \cup (A \cap B)$ and $(C \setminus A) \cup (C \setminus B)$, depicting clearly which area corresponds to the set, and how the set is constructed from A, B and C using the fundamental set operations. Use your Venn diagrams to conclude that the set-theoretic equality displayed above holds.

(b) Now prove the set-theoretic equality displayed above (for all sets A, B and C) using the algebra of sets (NB: the laws of the algebra of sets are on the last page of this exam).

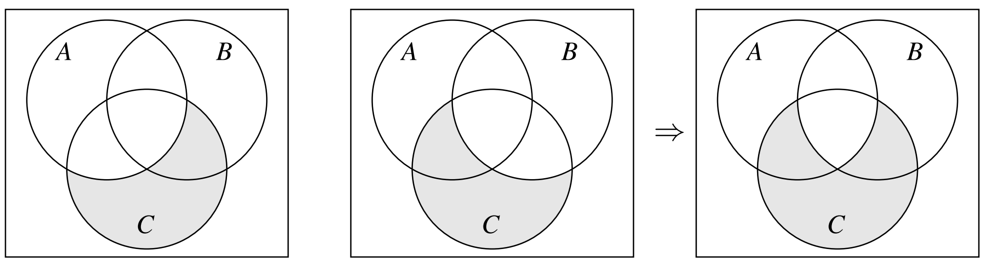
Solution:

(a) The following Venn diagrams depict the construction of the set $C' \cup (A \cap B)$:



C' $\quad A \cap B \quad$ $C' \cup (A \cap B)$

These Venn diagrams depict the construction of $(C \setminus A) \cup (C \setminus B)$:



$C \setminus A$ $\quad C \setminus B \quad$ $(C \setminus A) \cup (C \setminus B)$

The non-shaded area in the Venn-diagram of $C' \cup (A \cap B)$ corresponds exactly to the shaded area in the Venn-diagram of $(C \setminus A) \cup (C \setminus B)$. Therefore,

$$(C' \cup (A \cap B))' = (C \setminus A) \cup (C \setminus B)$$

(b) We derive the desired equality as follows:

$$\begin{aligned}
(C' \cup (A \cap B))' &= C'' \cap (A \cap B)' && \text{(De Morgan)} \\
&= C \cap (A \cap B)' && \text{(involution)} \\
&= C \cap (A' \cup B') && \text{(De Morgan)} \\
&= (C \cap A') \cup (C \cap B') && \text{(distributivity)} \\
&= (C \setminus A) \cup (C \setminus B) && \text{definition).}
\end{aligned}$$

6. Binary relations (4 + 4 + 5 points)
This task is about binary relations in the set $A := \{1, 2, 3, 6\}^2$.

(a) First we consider the equivalence relation $\equiv$ on A defined by the description

$$\langle a, b\rangle \equiv \langle c, d\rangle \qquad \Longleftrightarrow \qquad a \cdot b = c \cdot d.$$

Explicitly write down all equivalence classes (using the curly-bracket notation), and give a complete system of representatives for this equivalence relation.

(b) Next, we define a *strict* ordering relation $<$ on the set A by the description

$$\langle a, b\rangle < \langle c, d\rangle \qquad \Longleftrightarrow \qquad a \cdot b < c \cdot d.$$

Show that this relation is antisymmetric and transitive.

(c) Let $\leq$ be the *partial* order on A corresponding to the strict order $<$ introduced above. We now restrict our attention to the subset $B = \{2, 3, 6\}^2$ of A, that is,

$$B = \{\langle 2, 2\rangle, \langle 2, 3\rangle, \langle 2, 6\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle, \langle 3, 6\rangle, \langle 6, 2\rangle, \langle 6, 3\rangle, \langle 6, 6\rangle\}.$$

Use the algorithm you have learned to construct the Hasse diagram of the partial ordering relation $\leq$ on the set B. Explicitly write down the sets G_x and H_x obtained in the construction, and draw the Hasse diagram. (To simplify the notation, you can write 23 for $\langle 2, 3\rangle$, 33 for $\langle 3, 3\rangle$, and so on.)

Solution:

(a) The equivalence classes are

$$\begin{array}{lll} \{\langle 1,1\rangle\} & \{\langle 1,2\rangle, \langle 2,1\rangle\} & \{\langle 1,6\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 6,1\rangle\} \\ \{\langle 2,2\rangle\} & \{\langle 1,3\rangle, \langle 3,1\rangle\} & \\ \{\langle 3,3\rangle\} & \{\langle 2,6\rangle, \langle 6,2\rangle\} & \\ \{\langle 6,6\rangle\} & \{\langle 3,6\rangle, \langle 6,3\rangle\} & \end{array}$$

A complete system of representatives for this equivalence relation is therefore (for example)

$$\{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 6,6\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,6\rangle, \langle 3,6\rangle, \langle 1,6\rangle\}$$

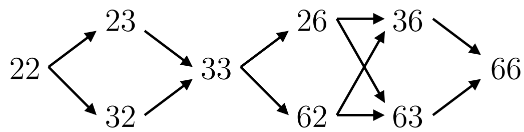
(b) (i) Suppose $\langle a, b\rangle \neq \langle c, d\rangle$. If $a \cdot b < c \cdot d$, then $c \cdot d \not< a \cdot b$. This shows that $\langle a, b\rangle < \langle c, d\rangle$ implies that it is not the case that $\langle c, d\rangle < \langle a, b\rangle$. Therefore, the proposed relation $<$ on A is antisymmetric.

(ii) If $a \cdot b < c \cdot d$ and $c \cdot d < e \cdot f$, then $a \cdot b < e \cdot f$. In other words, $\langle a, b\rangle < \langle c, d\rangle$ and $\langle c, d\rangle < \langle e, f\rangle$ together imply $\langle a, b\rangle < \langle e, f\rangle$. Hence, the proposed relation $<$ on A is transitive.

(c) The algorithm leads to the following sets G_x and H_x:

$$\begin{array}{ll} G_{22} = \{23, 26, 32, 33, 36, 62, 63, 66\} & H_{22} = \{23, 32\} \\ G_{23} = \{26, 33, 36, 62, 63, 66\} & H_{23} = \{33\} \\ G_{26} = \{36, 63, 66\} & H_{26} = \{36, 63\} \\ G_{32} = \{26, 33, 36, 62, 63, 66\} & H_{32} = \{33\} \\ G_{33} = \{26, 36, 62, 63, 66\} & H_{33} = \{26, 62\} \\ G_{36} = \{66\} & H_{36} = \{66\} \\ G_{62} = \{36, 63, 66\} & H_{62} = \{36, 63\} \\ G_{63} = \{66\} & H_{63} = \{66\} \\ G_{66} = \{\,\} & H_{66} = \{\,\} \end{array}$$

The Hasse diagram therefore looks like this:



7. Functions (4 + 4 + 4 points)

(a) In the set of all *People*, consider the two relations *IsFatherOf* and *IsOnlyChildOf* (where *Father* and *Child* are to be interpreted in the biological sense). For each of the relations *IsFatherOf*, *IsOnlyChildOf*, $IsFatherOf^{-1}$ and $IsOnlyChildOf^{-1}$, determine whether or not it is a function. Explain your answers.

For the remainder of this exercise, we work with the functions $sqr\colon \{x\colon x \geq 0\} \to \mathrm{R}$ and $sub\colon \mathrm{R} \to \mathrm{R}$ defined by

$$sqr(x) = x^2 \qquad\qquad sub(x) = x - 1.$$

(b) Give an explicit description (i.e. a formula for $f(x)$) of the function f given by

$$f := sub^{-1} \circ sqr^{-1} \circ sub,$$

and write down the domain of definition and range (image) of f.

(c) Express the function $g\colon \mathrm{R} \to \mathrm{R}$ defined by

$$g(x) := \sqrt{x^2 + 1}.$$

as a composition of the functions sqr, sub and their inverses.

Solution:

(a) Since there are fathers who have more than one child, *IsFatherOf* is not a function. However, every child has exactly one (biological) father, so $IsFatherOf^{-1}$ is a function.

A person who is an only child has two parents, so *IsOnlyChildOf* is not a function. However, a person can have at most one only child, so $IsOnlyChildOf^{-1}$ is a function.

(b) The function f is given by the description

$$f(x) = \sqrt{x - 1} + 1.$$

The domain of definition of this function is $D_f = \{x\colon x \geq 1\}$ and its range is $R_f = \{y\colon y \geq 1\}$.

(c) We have that $g = sqr^{-1} \circ sub^{-1} \circ sqr$.

8. Induction (10 points)

We claim that for all $n \in \mathrm{N}$

$$\sum_{k=0}^{n} 3^k = \frac{1}{2}(3^{n+1} - 1).$$

Your task is to prove this claim by mathematical induction.

NB: if we write out the sum, the claim is that

$$3^0 + 3^1 + 3^2 + \cdots + 3^n = \tfrac{1}{2}(3^{n+1} - 1) \qquad \text{for all } n \in \mathrm{N}.$$

Solution: Base case:

$$\sum_{k=0}^{0} 3^k = 3^0 = 1 = \frac{1}{2} \cdot 2 = \frac{1}{2}(3^1 - 1) = \frac{1}{2}(3^{0+1} - 1) \qquad \checkmark$$

Inductive hypothesis:

$$\sum_{k=0}^{m} 3^k = \frac{1}{2}(3^{m+1} - 1) \qquad (m \geq 0 \text{ arbitrary}).$$

Inductive step:

$$\begin{aligned}
\sum_{k=0}^{m+1} 3^k &= \Big(\sum_{k=0}^{m} 3^k\Big) + 3^{m+1} && \text{(by splitting the sum)} \\
&= \frac{1}{2}(3^{m+1} - 1) + 3^{m+1} && \text{(by the inductive hypothesis)} \\
&= \frac{1}{2}(3^{m+1} - 1) + \frac{1}{2} \cdot 2 \cdot 3^{m+1} && \text{(arithmetic)} \\
&= \frac{1}{2}(3^{m+1} + 2 \cdot 3^{m+1} - 1) && \text{(arithmetic)} \\
&= \frac{1}{2}(3 \cdot 3^{m+1} - 1) && \text{(arithmetic)} \\
&= \frac{1}{2}(3^{(m+1)+1} - 1) && \text{(arithmetic)} \qquad \checkmark
\end{aligned}$$

This is exactly the statement in the claim for $n = m+1$, which completes the proof.